Example 1.1. The chart below shows historical data for high and low temperatures in Novato, CA for Aug. 20th for the past 5 years.

<table>
<thead>
<tr>
<th>high temperature</th>
<th>76</th>
<th>80</th>
<th>75</th>
<th>77</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td>low temperature</td>
<td>58</td>
<td>57</td>
<td>57</td>
<td>50</td>
<td>46</td>
</tr>
</tbody>
</table>

This is an example of a relation. We’ll give the relation between high and low temperatures a name, and call it \( g \).

Definition 1.2. A relation is a correspondence between input numbers (x-values) and output numbers (y-values).

Question 1.3. If we let \( x \) be the high temperatures and \( y \) be the low temperatures, what \( y \)-value corresponds to the \( x \) value of 76? __________

What \( y \) value corresponds to the \( x \) value of 80? __________

Note 1.4. It is possible to draw a relation as a diagram with arrows that go from \( x \) values to \( y \) values. Draw three more arrows to complete the diagram for \( g \) below, based on the table of temperatures above.

![Diagram](image)

Definition 1.5. The domain of a relation is all possible \( x \)-values (input values). The range of a relation is all possible \( y \)-values (output values).

Question 1.6. What is the domain of the relation \( g \)? __________

What is the range of the relation \( g \)? __________
Example 1.7. If we add a row to show the year, we can introduce a new relation between year and high temperature for August 20th for that year. Call this relation \( h \).

<table>
<thead>
<tr>
<th>year</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>high temperature</td>
<td>76</td>
<td>80</td>
<td>75</td>
<td>77</td>
<td>75</td>
</tr>
<tr>
<td>low temperature</td>
<td>58</td>
<td>57</td>
<td>57</td>
<td>50</td>
<td>46</td>
</tr>
</tbody>
</table>

Example 1.8. Draw a diagram for the relation \( h \), where \( x \) is the year and \( y \) is the high temperature.

Definition 1.9. A function is a relation such that each input value is assigned to one and only one output value.

Question 1.10. Of the two relations \( g \) and \( h \), which one is a function? __________ Why?

Notation 1.11. The notation \( y = f(x) \) is used to represent a function.

\( f \) is the name of the function.
\( x \) represents the input (also called an independent variable, or the “argument”).
\( y \) represents the output, also called the dependent variable.
\( f(x) \) also represents the output. Sometimes it is handy to think of \( f(x) \) as another word for \( y \).

Question 1.12. For the function \( y = h(x) \) above, (circle one) \( year / high \ text{temperature} / low \ text{temperature} \) is the independent variable and (circle one) \( year / high \ text{temperature} / low \ text{temperature} \) is the dependent variable.
Question 1.13. For $y = h(x)$, what is $h(2008)$? __________

For what $x$ value(s) is $h(x) = 75$? __________

Note 1.14. Another way to describe a function (or a relation) is as a set of ordered pairs.

Example 1.15. The function $h(x)$ can be described as


Example 1.16. Write the relation $g(x)$ as a set of ordered pairs.

Note 1.17. Another way to describe a function (or a relation) is with a graph.

Example 1.18. Draw the relation $g(x)$ as a graph below. Hint: plot the ordered pairs!
Example 1.19. Consider the graph of the function \( y = f(x) \) below.

![Graph of \( y = f(x) \)](image)

**Question 1.20.** What is \( f(2) \)?

What is \( f(5) \)?

For what \( x \) values is \( f(x) = 1 \)?

**Definition 1.21.** The **y-intercept(s)** of a graph are the \( y \)-value(s) where the graph crosses the \( y \)-axis. The **x-intercepts** are the \( x \)-values where the graph ________________.

**Question 1.22.** What are the y-intercepts and x-intercepts for the graph of the function \( y = f(x) \) above?

Example 1.23. Which of the following equations describes the function \( f(x) \) graphed above?

a) \( f(x) = x^2 - 3 \)

b) \( f(x) = (x + 1)(x - 2) - 3 \)

c) \( f(x) = x^2 - 4x + 1 \)

d) \( f(x) = 3(x - 2) \)

**Question 1.24.** Based on the equation you chose, what is \( f(-2) \)? ___ \( f(100) \)? ___ Do these answers seem plausible based on the graph?
Question 1.25. Which of the following graphs represent functions?

Fact 1.26. The Vertical Line Test says: If there is a vertical line that intersects the graph in more than one point, then the graph (circle one) is / is not the graph of a function. Otherwise, the graph is / is not the graph of a function.

Example 1.27. List all the different ways of describing a relation (or a function) that have been mentioned so far.

1.

2.

3.

4.

5.
Example 1.28. Consider the following graph of a relation $p$.

a) Write the relation as a set of ordered pairs.
b) Write it as a table of values.
c) Find its domain.
d) Find its range.
e) Is $p$ a function?
f) What is $p(3)$?
g) For what $x$ values is $p(x) = 2$?
2 Class 2 - §1.2 and 1.3 - Interval Notation and Linear Functions

Notation 2.1. Page 12 of the textbook gives a chart detailing how to describe intervals using set notation (with curly brackets \( \{ \} \)), a number line, and interval notation.

Example 2.2. Complete the following chart by shading in the appropriate parts of the number line and filling in the interval notation. \( a \) and \( b \) are real numbers.

<table>
<thead>
<tr>
<th>Name</th>
<th>Set Notation</th>
<th>Number Line</th>
<th>Interval Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>open interval</td>
<td>( { x \mid a &lt; x &lt; b } )</td>
<td>![Number Line](open Interval)</td>
<td>((a, b))</td>
</tr>
<tr>
<td>closed interval</td>
<td>( { x \mid a \leq x \leq b } )</td>
<td>![Number Line](closed Interval)</td>
<td>([a, b])</td>
</tr>
<tr>
<td>half open interval</td>
<td>( { x \mid a &lt; x \leq b } )</td>
<td>![Number Line](half Open Interval)</td>
<td></td>
</tr>
<tr>
<td>half open interval</td>
<td>( { x \mid a \leq x &lt; b } )</td>
<td>![Number Line](half Open Interval)</td>
<td></td>
</tr>
<tr>
<td>unbounded interval</td>
<td>( { x \mid a &lt; x } )</td>
<td>![Number Line](unbounded Interval)</td>
<td></td>
</tr>
<tr>
<td>unbounded interval</td>
<td>( { x \mid a \leq x } )</td>
<td>![Number Line](unbounded Interval)</td>
<td></td>
</tr>
<tr>
<td>unbounded interval</td>
<td>( { x \mid x &lt; b } )</td>
<td>![Number Line](unbounded Interval)</td>
<td></td>
</tr>
<tr>
<td>unbounded interval</td>
<td>( { x \mid x \leq b } )</td>
<td>![Number Line](unbounded Interval)</td>
<td></td>
</tr>
<tr>
<td>all real numbers</td>
<td>( \mathbb{R} )</td>
<td>![Number Line](all Real Numbers)</td>
<td></td>
</tr>
</tbody>
</table>
### Example 2.3. Translate between set notation and interval notation.

<table>
<thead>
<tr>
<th>Set Notation</th>
<th>Interval Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( { x \mid -3 \leq x &lt; 1 } )</td>
<td></td>
</tr>
<tr>
<td>2. ( (\infty, 5) )</td>
<td></td>
</tr>
<tr>
<td>3. ( { x \mid -15 &lt; x } )</td>
<td></td>
</tr>
<tr>
<td>4. ( { x \mid x \geq -2 } )</td>
<td></td>
</tr>
<tr>
<td>5. ( (3, 17] )</td>
<td></td>
</tr>
<tr>
<td>6. ( [42, \infty) )</td>
<td></td>
</tr>
<tr>
<td>7. ( { x \mid 4 \geq x &gt; 0 } )</td>
<td></td>
</tr>
</tbody>
</table>

**Note 2.4.** In interval notation, the smaller number is always on the left side.

**Note 2.5.** When translating from set notation it is helpful to first rewrite any expressions with > or ≥ signs, using equivalent expressions with < and ≤ signs. For example, if you rewrite \( \{ x \mid 4 \geq x > 0 \} \) as \( \{ x \mid 0 < x \leq 4 \} \), that makes it easier to translate to interval notation, because the smaller number is already on the left side.
Example 2.6. Find the domain and range for the function graphed below. Write the domain and range in both set notation and in interval notation.

Example 2.7. Find the domain and range for the function below and write your answers in interval notation. The arrow means that the graph keeps on going.
Example 2.8. Find the domain and range for the function below and write your answers in interval notation.

![Graph of a function](image)

Example 2.9. In July, 2010, I tried to sell our 2004 Chevy Aveo. At the time it was worth $4,800 (Blue Book value), but each month its value went down by $70.

a) Write a function to describe its value over time.

b) Graph it on your calculator.

c) If the car continues to depreciate at this value, what should it be worth now? (Would anyone like to make me an offer?)

d) According to this model, when will its value be zero?
Definition 2.10. A *linear function* is a function that can be written in the form \( y = mx + b \). \( m \) represents the slope (rise over run), and \( b \) represents the y-intercept (where the line crosses the y-axis).

Example 2.11. In the car example,

a) what is the slope, and what are the units?

b) what is the y-intercept and what are the units?

c) what is the x-intercept and what are the units?

Definition 2.12. The y-intercept is __________________________.
The x-intercept is __________________________.

Note 2.13. To find the y-intercept algebraically from the equation for a line, plug in ______ for \( x \) and solve for \( y \). To find the x-intercept, plug in ______ for \( y \) and solve for \( x \).

Example 2.14. A nuclear energy plant in Hanford, Washington leaked radiation in the the Columbia river. Cancer deaths downstream in Oregon were linked to the level of radiation exposure.

<table>
<thead>
<tr>
<th>County</th>
<th>Radiation Index</th>
<th>Deaths per 100,000 residents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wasco</td>
<td>1.6</td>
<td>138</td>
</tr>
<tr>
<td>Clatsop</td>
<td>8.3</td>
<td>210</td>
</tr>
</tbody>
</table>
a) Assume that there is a linear relationship between radiation exposure and cancer deaths and describe it with an equation.

b) What is the y-intercept and what does it mean?

c) What is the slope and what does it mean?
3 Class 3 - Sections 1.3, 1.4, and 1.5

Example 3.1. Find an equation for the line graphed below.

![Graph of a line]

Example 3.2. Find equations for the two lines graphed below.

![Graph of two lines]

Example 3.3. Which of these equations represent lines?

1. \( y = 5x - 13.6 \)
2. \( 4y + 3x = 17 \)
3. \( y = 13x^2 + 7 \)
4. \( y - 19.2 = 3.5(x - 2.1) \)
5. \( 2y = \frac{4}{x} + 7 \)
For each equation above that represents a line, find the slope and y-intercept for the line.
**Fact 3.4.** Two lines are parallel if their slopes are \[ \text{__________} \].

**Example 3.5.** Find the equation for a line through the point \((2, -1)\) that is parallel to the line \(3y + 2x = 12\). Draw both lines on the graph below.

![Graph showing two lines](image)

**Fact 3.6.** Two lines are perpendicular if their slopes are \[ \text{__________} \].

**Example 3.7.** For each slope of a line given in the left column, write down the slope of a line that is perpendicular to it in the right column.

<table>
<thead>
<tr>
<th>(m_1)</th>
<th>(m_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4 (\frac{4}{7})</td>
<td></td>
</tr>
<tr>
<td>-5</td>
<td></td>
</tr>
<tr>
<td>-(\frac{1}{10})</td>
<td></td>
</tr>
</tbody>
</table>
Example 3.8. Find the equation for a line through the point \((2, -1)\) that is perpendicular to the line \(3y + 2x = 12\), and draw both lines on the graph below.

Example 3.9. Is the following figure a perfect rectangle? Use what you know about parallel and perpendicular lines to prove your answer!
Example 3.10. Three kids want to start a snow cone business. The shaved ice machine costs $29.99. Syrup costs about 12 cents per snow cone and the paper cones cost 7 cents each.

a) Write an equation for the cost $C(x)$ for making $x$ snow cones. Include the start up cost for buying the shaved ice machine and the per cone cost of syrup and paper.

b) Suppose the kids charge 75 cents per snow cone. Write an equation for the revenue $R(x)$ as a function of the number of snow cones that they sell.

c) Write an equation to help answer the question: How many snow cones do the kids have to sell to break even?
There are several ways to solve an equation like the cost-revenue equation that you just wrote.

1. Algebraic method: Use algebra

2. Graphical method 1: Use your calculator to graph the left and right sides of the equation and look for the point where ________________.

3. Graphical method 2: Rewrite the equation by putting everything on the left side, and then graph the left side and look for the point where ________________.  


Example 3.11. Solve for \( x \) using a graphical method and check your answer using algebra.

\[ 3x - 7.5 = 0.2x - 4 \]

Example 3.12. Solve for \( x \) using a graphical method and check your answer using algebra.

\[ -2x + 15 = -2(x + 6) \]

Example 3.13. Solve for \( x \) using a graphical method and check your answer using algebra.

\[ 3(2x - 8) + 6 = 6(x - 3) \]
Example 3.14. (If there is extra time) Solve for $x$ using a graphical method:

\[
\frac{x + 1}{6} + \frac{2x - 8}{2} = -4
\]

Note 3.15. An equation for which there is no solution is called a \textit{contradiction}. When you graph both sides of a contradictory linear equation, you get two lines that ________________.

An equation which holds for all real numbers is called an \textit{identity}. When you graph both sides of a linear equation that is an identity, you get two lines that ________________.

An equation holds for some values of $x$ but not for others is called a \textit{conditional equation}. When you graph both sides of a conditional linear equation, you get two lines that ________________.
Example 4.1. Sprint has a cell phone plan that costs $49.99 per month and includes unlimited texting and 450 minutes of phone calls per month. Additional minutes cost $0.45 per minute. LG’s phone plan costs $59.90 pre month, and also includes unlimited texting and 450 minutes of phone calls per month. Additional minutes in LG’s plan cost $0.35 per month.

a) Write a function \( s(x) \) to describe the cost for Sprint’s plan as a function of \( x \), where \( x \) is the number of minutes over 450.

b) Write a function \( \ell(x) \) to describe the cost for LG’s plan as a function of \( x \), the number of minutes over 450.

c) Graph the two functions on your calculator and sketch the graph below.

d) Find the intersection point on your graph. What are its coordinates and what does this mean?

e) Based on the graph, for what values of \( x \) is Sprint’s plan cheaper than LG’s plan?

f) For what values of \( x \) is Sprint’s plan more expensive than LG’s plan?
g) For what values of $x$ is $s(x) < \ell(x)$?

h) For what values of $x$ is $s(x) > \ell(x)$?

i) Solve the inequality $s(x) < \ell(x)$ algebraically.

Example 4.2. In the graph below, label the solid line $f(x)$ and the dashed line $g(x)$.

![Graph showing solid and dashed lines]

a) Is $f(5) < g(5)$?
b) Is $f(4) < g(4)$?
c) Is $f(3) < g(3)$?
d) Is $f(-10) < g(-10)$?
e) Is $f(3.7) < g(3.7)$?
f) For what values of $x$ is $f(x) < g(x)$?
Example 4.3. For the graph below, label the solid line $y_1$ and the dashed line $y_2$.

For what values of $x$ is $y_2 \geq y_1$?
For what values of $x$ is $y_2 < y_1$?

Example 4.4. There are several ways to solve the inequality

$$6x + 8 < -2x + 5$$

1. Algebraic Method - Use algebra:
2. Graphical Method 1: Graph the left and right side of the inequality and using the intersection point. Sketch the graph below and circle the part of the graph that you are interested in (that corresponds to the solution of the inequality).

Describe the x-values that solve the inequality in interval notation: __________

3. Graphical Method 2: Rewrite the inequality by moving everything to the left to get the inequality: ____________________.

Now, graph the left side of the rewritten inequality. Circle the part of the graph that you are interested in (that corresponds to the solution of the inequality).

Describe the x-values that solve the inequality in interval notation: __________
Example 4.5. Use a graphical method to solve the inequality

\[ 5x - 2 \geq 2x - 6 \]

Example 4.6. Use a graphical method to solve the inequality

\[ -7 < 3x + 5 \]
Definition 4.7. We say that the variable $y$ varies directly with $x$ if $y = k \cdot x$ for some number $k$ ($k \neq 0$).

We say that the variable $y$ varies inversely with $x$ if $y = \frac{k}{x}$ for some number $k$ ($k \neq 0$).

$k$ is called the constant of variation.

Example 4.8. In 1.5 hours, Joe can bike 16 miles. Assume distance is directly proportional to time, that is, $y = kx$ for some number $k$, where $y$ represents distance traveled and $x$ represents time. Find the constant of proportionality $k$. How far can Joe bike in 2 hours?

Example 4.9. A movie transfers in 48 minutes at a speed of 256 kB/sec. How long will it take to transfer the same movie at 1000 kB/sec? Set up an equation using inverse variation and solve for the constant of variation first, then answer the question.
5 Class 5 - Section 2.1 - Symmetry of Functions

Example 5.1. What kind of symmetry do you notice in the graph of the function $g(x) = x^2 + 1$ below?

Describe this symmetry in words.

Each point on the right side of the graph has a mirror image point on the left side. On the graph, draw the mirror image for each the following points and label each mirror image point with its coordinates:

- $(2, 5)$
- $(1, 2)$
- $(a, b)$
- $(x, f(x))$

This is an example of a graph that is symmetric with respect to the y-axis.

Definition 5.2. A graph is called symmetric with respect to the y-axis if for every point $(a, b)$ on the graph, the point $(b, a)$ is also on the graph.

Definition 5.3. A function is called even if its graph is symmetric with respect to the y-axis.
Example 5.4. Compare the following values for the even function $g(x) = x^2 + 1$ graphed above:

\[ g(1) = \quad g(-1) = \]

\[ g(2) = \quad g(-2) = \]

\[ g(3) = \quad g(-3) = \]

Question 5.5. For an even function $f$, how do the values of $f(x)$ and $f(-x)$ compare?
Example 5.6. What kind of symmetry do you see in the graph of $h(x) = \frac{x^3}{64}$ drawn below?

Describe this symmetry in words.

Each point on the right side of the graph has a corresponding, symmetric point on the left side. Draw the symmetric points for the following points on the graph and label the symmetric points with their coordinates:

- $(4, 1)$
- $(8, 8)$
- $(a, b)$
- $(x, f(x))$

This is an example of a graph that is symmetric with respect the origin.

Definition 5.7. A graph is called symmetric with respect to the origin if for every point $(a, b)$ on the graph, the point ______ is also on the graph.

Definition 5.8. A function is called odd if its graph is symmetric with respect to the origin.
Example 5.9. Compare the following values for the odd function \( h(x) = \frac{x^3}{64} \) graphed above:

\[
\begin{align*}
  h(1) &= \underline{\phantom{0}} \\
  h(-1) &= \underline{\phantom{0}} \\
  h(2) &= \underline{\phantom{0}} \\
  h(-2) &= \underline{\phantom{0}} \\
  h(3) &= \underline{\phantom{0}} \\
  h(-3) &= \underline{\phantom{0}} \\
  h(4) &= \underline{\phantom{0}} \\
  h(-4) &= \underline{\phantom{0}}
\end{align*}
\]

Question 5.10. For an odd function \( f \), how do the values of \( f(x) \) and \( f(-x) \) compare?

Example 5.11. The relation graphed below has a different kind of symmetry.

Describe this symmetry in words.

The mathematical term for this kind of symmetry is \textit{symmetric with respect to the x-axis}.

Definition 5.12. A graph is called \textit{symmetric with respect to the x-axis} if for every point \((a, b)\) on the graph, the point \(\underline{\phantom{0}}\) is also on the graph.
Example 5.13. Fill in the following chart of equivalent terms.

<table>
<thead>
<tr>
<th>Name</th>
<th>Type of Symmetry</th>
<th>Math Word for Symmetry</th>
<th>Algebraic Def</th>
<th>Another Algebraic Def</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>mirror symmetry</td>
<td>symmetric with respect</td>
<td>if ((a, b)) is on the graph, (f(-x) = )</td>
<td>so is (f(-x) = )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(y-axis is the mirror line)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>odd</td>
<td>180° rotational symmetry</td>
<td>symmetric with respect</td>
<td>if ((a, b)) is on the graph, (f(-x) = )</td>
<td>so is (f(-x) = )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(around the origin)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N/A</td>
<td>mirror symmetry</td>
<td>symmetric with respect</td>
<td>if ((a, b)) is on the graph, (f(-x) = )</td>
<td>so is (f(-x) = )</td>
<td>N/A</td>
</tr>
<tr>
<td></td>
<td>(y-axis is the mirror line)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Question 5.14. Are all functions either even or odd? Can you think of a function that is neither even nor odd? both even and odd?

Example 5.15. Which of the following functions are even? odd? neither?

a) \(f(x) = -x^4 + 2x^2 + 2\)

b) \(h(x) = 3x^3 - 8x\)

c) \(g(x) = 3x^3 - 8x + 3\)

d) \(g(x) = x^2 - 4x\)

e) \(f(x) = \frac{1}{x}\)

Question 5.16. Why do you think functions are called even and odd? Hint: think about the exponents in the previous examples.
Example 5.17. Which of the following two functions is increasing and which is decreasing?

Definition 5.18. (Informal Definition) A function is **increasing** if the y-values go (circle one) up / down as the x-values go up. That is, the y-values go (circle one) up / down as you traverse the function from left to right.

A function is **decreasing** if the y-values go (circle one) up / down as the x-values go up. That is, the y-values go (circle one) up / down as you traverse the function from left to right.

Example 5.19. Look at the function graphed below. Where is it increasing and where is it decreasing? Describe the places in terms of the x-values and use interval notation.

increasing: __________________   decreasing: __________________
Definition 5.20. (Informal Definition) A function is continuous if you can draw it without picking up your pencil.

Example 5.21. Which of these functions are continuous?

Note 5.22. Even though the middle function is not continuous, it is still continuous on certain intervals. Always describe the intervals in terms of the x-values. For example, the function is continuous on the interval $(-\infty, 0)$. What other interval is it continuous on?

Note 5.23. Whenever you are describing parts of a function, always refer to the x-values. The only exception is when you are talking about the range, which you should describe in terms of y-values.
6 Classes 6 and 7 - Section 2.2 and 2.3 - Transformations of Functions

As a warm-up, we need to practice using function notation.

Example 6.1. Define $g(x) = \sqrt{x}$. Then we can rewrite $g(x) + 6$ as $\sqrt{x} + 6$. Rewrite the following. Be careful about what goes inside the square root sign and what goes outside.

a) $g(x) - 2 = \underline{\hspace{2cm}}$

b) $g(x - 2) = \underline{\hspace{2cm}}$

c) $g(3x) = \underline{\hspace{2cm}}$

d) $3g(x) = \underline{\hspace{2cm}}$

We can rewrite the expression $\sqrt{x} + 17$ in terms of $g(x)$ as $g(x) + 17$. This works because $g(x) = \sqrt{x}$. Rewrite the following functions in terms of $g(x)$:

e) $\sqrt{x} - 4 = \underline{\hspace{2cm}}$

f) $\sqrt{x + 12} = \underline{\hspace{2cm}}$

g) $-36 \cdot \sqrt{x} = \underline{\hspace{2cm}}$

h) $\sqrt{\frac{1}{2}x} = \underline{\hspace{2cm}}$

Example 6.2. Define $h(x) = x^2$. Rewrite the following expressions. Do not simplify.

a) $h(x - 7) = \underline{\hspace{2cm}}$

b) $h(x) - 42 = \underline{\hspace{2cm}}$

c) $h(3.5x) + 4 = \underline{\hspace{2cm}}$

d) $\frac{1}{2}h(x + 2) = \underline{\hspace{2cm}}$

Rewrite the following expressions in terms of $h(x)$. The function $h(x)$ is still $x^2$.

e) $3x^2 = \underline{\hspace{2cm}}$

f) $(x - \frac{7}{2})^2 = \underline{\hspace{2cm}}$

g) $x^2 + 5 = \underline{\hspace{2cm}}$

h) $(5x)^2 = \underline{\hspace{2cm}}$

i) $-3(x - 2)^2 + 7 = \underline{\hspace{2cm}}$
Question 6.3. For a number $c$ and a function $f(x)$, how does the graph of $y = f(x) + c$ compare to the graph of $y = f(x)$?

To answer this question, we’ll work an example where $f(x) = |x|$.

Rewrite the following functions in terms of $f(x)$.

\[
\begin{align*}
y_1 &= |x| \quad \Rightarrow \quad y_1 = f(x) \\
y_2 &= |x| + 5 \quad \Rightarrow \quad y_2 = \text{__________} \\
y_3 &= |x| - 1 \quad \Rightarrow \quad y_3 = \text{__________} \\
y_4 &= |x| + 3 \quad \Rightarrow \quad y_4 = \text{__________} \\
\end{align*}
\]

Graph $y_1, y_2, y_3,$ and $y_4$ and sketch their graphs below.

Question 6.4. If $c > 0$, how does the graph of $y = f(x) + c$ compare with the graph of $y = f(x)$?

Question 6.5. If $c > 0$, how does the graph of $y = f(x) - c$ compare with the graph of $y = f(x)$?

Conclusion: If $c > 0$, then the graph of $y = f(x) + c$ is obtained from the graph of $y = f(x)$ by _______________________________. The graph of $y = f(x) - c$ is obtained from the graph of $y = f(x)$ by _______________________________.
**Question 6.6.** For a number \( c \) and a function \( f(x) \), how does the graph of \( y = f(x + c) \) compare to the graph of \( y = f(x) \)?

To answer this question, we’ll work an example where \( f(x) = x^2 \).

Rewrite the following functions in terms of \( f(x) \).

\[
\begin{align*}
y_1 &= x^2 \quad \Rightarrow \quad y_1 = f(x) \\
y_2 &= (x - 1)^2 \quad \Rightarrow \quad y_2 = f(x)
\end{align*}
\]

\[
\begin{align*}
y_3 &= (x - 3)^2 \quad \Rightarrow \quad y_3 = f(x)
\end{align*}
\]

\[
\begin{align*}
y_4 &= (x + 4)^2 \quad \Rightarrow \quad y_4 = f(x)
\end{align*}
\]

Graph \( y_1, y_2, y_3, \) and \( y_4 \) and sketch their graphs below.

**Question 6.7.** If \( c > 0 \), how does the graph of \( y = f(x + c) \) compare with the graph of \( y = f(x) \)?

**Question 6.8.** If \( c > 0 \), how does the graph of \( y = f(x - c) \) compare with the graph of \( y = f(x) \)?

**Conclusion:** If \( c > 0 \), then the graph of \( y = f(x+c) \) is obtained from the graph of \( y = f(x) \) by ______________________________. The graph of \( y = f(x-c) \) is obtained from the graph of \( y = f(x) \) by ______________________________.
Question 6.9. For a number $c$ and a function $f(x)$, how does the graph of $y = cf(x)$ compare to the graph of $y = f(x)$?

To answer this question, we’ll work an example where $f(x) = \sin(x)$.

Rewrite the following functions in terms of $f(x)$.

$$
\begin{align*}
y_1 &= \sin(x) \quad \Rightarrow \quad y_1 = f(x) \\
y_2 &= 3 \sin(x) \quad \Rightarrow \quad y_2 = \underline{\phantom{f(x)}} \\
y_3 &= 2 \sin(x) \quad \Rightarrow \quad y_3 = \underline{\phantom{f(x)}} \\
y_4 &= \frac{1}{2} \sin(x) \quad \Rightarrow \quad y_4 = \underline{\phantom{f(x)}}
\end{align*}
$$

Graph $y_1$, $y_2$, $y_3$, and $y_4$ and sketch their graphs below.

Question 6.10. If $c > 1$, how does the graph of $y = cf(x)$ compare with the graph of $y = f(x)$?

Question 6.11. If $c < 1$, how does the graph of $y = cf(x)$ compare with the graph of $y = f(x)$?

Conclusion: If $c > 1$, then the graph of $y = cf(x)$ is obtained from the graph of $y = f(x)$ by (circle one) stretching / shrinking the graph (circle one) horizontally / vertically by a factor of \underline{\phantom{1}}.

If $c < 1$, the graph of $y = cf(x)$ is obtained from the graph of $y = f(x)$ by (circle one) stretching / shrinking the graph (circle one) horizontally / vertically by a factor of \underline{\phantom{1}}.
**Question 6.12.** For a number \( c \) and a function \( f(x) \), how does the graph of \( y = f(cx) \) compare to the graph of \( y = f(x) \)?

To answer this question, we’ll work an example where \( f(x) = \sin(x) \).

Rewrite the following functions in terms of \( f(x) \).

\[
\begin{align*}
y_1 &= \sin(x) \quad \Rightarrow \quad y_1 = f(x) \\
y_2 &= \sin(2x) \quad \Rightarrow \quad y_2 = f(2x) \\
y_3 &= \sin(3x) \quad \Rightarrow \quad y_3 = f(3x) \\
y_4 &= \sin\left(\frac{1}{2}x\right) \quad \Rightarrow \quad y_4 = f\left(\frac{1}{2}x\right)
\end{align*}
\]

Graph \( y_1, y_2, y_3, \) and \( y_4 \) and sketch their graphs below.

**Question 6.13.** If \( c > 1 \), how does the graph of \( y = f(cx) \) compare with the graph of \( y = f(x) \)?

**Question 6.14.** If \( c < 1 \), how does the graph of \( y = f(cx) \) compare with the graph of \( y = f(x) \)?

**Conclusion:** If \( c > 1 \), then the graph of \( y = f(cx) \) is obtained from the graph of \( y = f(x) \) by (circle one) stretching / shrinking the graph (circle one) horizontally / vertically by a factor of \( c \).

If \( c < 1 \), the graph of \( y = f(cx) \) is obtained from the graph of \( y = f(x) \) by (circle one) stretching / shrinking the graph (circle one) horizontally / vertically by a factor of \( \frac{1}{c} \).
Question 6.15. How do the graphs of \( y = -f(x) \) and \( y = f(-x) \) compare to the graph of \( y = f(x) \)?

To answer this question, we’ll work an example where \( f(x) = \sqrt{x} \).

Rewrite the following functions in terms of \( f(x) \).

\[
\begin{align*}
y_1 &= \sqrt{x} \quad \Rightarrow \quad y_1 = f(x) \\
y_2 &= -\sqrt{x} \quad \Rightarrow \quad y_2 = \underline{\phantom{0}} \\
y_3 &= \sqrt{-x} \quad \Rightarrow \quad y_3 = \underline{\phantom{0}} \\
y_4 &= -\sqrt{-x} \quad \Rightarrow \quad y_4 = \underline{\phantom{0}} \\
\end{align*}
\]

Graph \( y_1, y_2, y_3, \) and \( y_4 \) and sketch their graphs below.

Question 6.16. How does the graph of \( y = -f(x) \) compare with the graph of \( y = f(x) \)?

Question 6.17. How does the graph of \( y = f(-x) \) compare with the graph of \( y = f(x) \)?

Question 6.18. How does the graph of \( y = -f(-x) \) compare with the graph of \( y = f(x) \)?

Conclusion: The graph of \( y = -f(x) \) is obtained from the graph of \( y = f(x) \) by \underline{\phantom{0}}. The graph of \( y = f(-x) \) is obtained from the graph of \( y = f(x) \) by \underline{\phantom{0}}.
Example 6.19. Give the equation of each function graphed below.

![Graph](image1.png)

Equation:  

Equation:  

Example 6.20. Give the equation of each function graphed below.

![Graph](image2.png)

Equation:  

Equation:
Example 6.21. Describe how the graph of \( y = |x + 2| - 6 \) would be obtained by translating the graph of \( y = |x| \). Sketch both graphs on the same axes below.

The graph of \( y = |x + 2| - 6 \) is shifted \( \_\_\_\_\_\_\_\_\_\_\_\_ \) units \( (\text{circle one}) \) left/ right and \( \_\_\_\_\_\_\_\_\_\_\_\_ \) units \( (\text{circle one}) \) up/ down.

Example 6.22. What is the equation for the cubing function \( x^3 \),

- shifted up by 4 units?
- shifted right by 5 units?
- shifted up by 4 units and right by 5 units?

Example 6.23. What is the equation of the squaring function

- stretched vertically by a factor of 5?
- stretched horizontally by a factor of 2?
- shrunk vertically by a factor of \( \frac{1}{3} \)?
- shrunk horizontally by a factor of \( \frac{1}{4} \)

Example 6.24. How does the graph of \( g(x) = -\sqrt[3]{x} \) compare to the graph of \( f(x) = \sqrt[3]{x} \)?
Example 6.25. Find the equations for the following four graphs.
Example 6.26. The graph of \( y = g(x) \) is drawn below.

Graph the following functions:

- \( y = -f(x) \)
- \( y = f(2x) \)
- \( y = f(x - 1) \)
- \( y = 2f(x) - 1 \)
6 CLASSES 6 AND 7 - SECTION 2.2 AND 2.3 - TRANSFORMATIONS OF FUNCTIONS
Summary:

- Numbers on the outside of the function affect the y-values and result in vertical motions. These motions are in the direction you expect.

- Numbers on the inside of the function affect the x-values and result in horizontal motions. These motions go in the opposite direction from what you expect.

- Adding results in a shift (translations)

- Multiplying results in a stretch or shrink

- A negative sign results in a reflection

<table>
<thead>
<tr>
<th></th>
<th>Addition / Subtraction</th>
<th>Multiplication / Division</th>
<th>Negative Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Shifting</td>
<td>Stretches or Shrinks</td>
<td>Reflecting</td>
</tr>
<tr>
<td>Number outside of the function</td>
<td>$f(x) + c$</td>
<td>$cf(x)$</td>
<td>$-f(x)$</td>
</tr>
<tr>
<td>affects the y-values</td>
<td>Vertical shift</td>
<td>Vertical stretch (if $c &gt; 1$)</td>
<td>Reflection across the x-axis</td>
</tr>
<tr>
<td>vertical motion</td>
<td></td>
<td>Vertical shrink (if $0 &lt; c &lt; 1$)</td>
<td></td>
</tr>
<tr>
<td>Number inside of the function</td>
<td>$f(x + c)$</td>
<td>$f(cx)$</td>
<td>$f(-x)$</td>
</tr>
<tr>
<td>affects the x-values</td>
<td>Horizontal shift</td>
<td>Horizontal shrink (if $c &gt; 1$)</td>
<td>Reflection across the y-axis</td>
</tr>
<tr>
<td>horizontal motion</td>
<td></td>
<td>Horizontal stretch (if $0 &lt; c &lt; 1$)</td>
<td></td>
</tr>
</tbody>
</table>

Note 6.27. Horizontal motions go in the opposite direction from what you expect.

Example 6.28. If time, try the following applet:
http://home.earthlink.net/fossmountdesign/Applets/AbsoluteApplet.html
There is no 7th section because classes 6 and 7 were combined!
8 Class 8 - Section 2.6 - Operations on Functions

Example 8.1. The function \( f(x) = 19.95 + 0.69x \) gives the cost of renting a cargo van at U-haul for a day, where \( x \) represents the number of miles driven. The function \( g(x) = 29.95 + 0.99x \) gives the cost of renting a 14-foot truck as a function of \( x \), the miles driven.

**Notation 8.2.** The notation \((f + g)(x)\) means to add the functions \( f \) and \( g \) together. In other words:

\[
(f + g)(x) = f(x) + g(x) = (19.95 + 0.69x) + (29.95 + 0.99x) = 19.95 + 0.69x + 29.95 + 0.99x = 49.90 + 1.68x
\]

**Question 8.3.** What could the function \((f + g)(x)\) mean in the context of this problem?

**Notation 8.4.** The notation \((f - g)(x)\) means to subtract the function \( g \) from the function \( f \). In other words:

\[
(f - g)(x) = f(x) - g(x)
\]

Write an equation for \((g - f)(x)\), where \( f \) and \( g \) are the same functions as before.

What is \((g - f)(55)\)?

What does the function \((g - f)(x)\) mean in the context of the U-haul rental problem?

Example 8.5. Suppose the function \( f(x) = 400 + x^2 \) represents the number of computers sold by Joe’s Fictional Computer Store, where \( x \) is the number of years since the year 2000. The function \( g(x) = 1200 - 25x \) represents the average price of a computer sold in the store in dollars, where \( x \) is still the number of years since 2000.

**Notation 8.6.** The notation \((f g)(x)\) (also written \( f \cdot g)(x)\)) means to multiply the functions \( f \) and \( g \) together. In other words:

\[
(f g)(x) = f(x) \cdot g(x)
\]

Write an equation for \((f g)(x)\).
What is \((fg)(15)\)? What are the units and what does this number represent?

**Example 8.7.** Suppose the function \(f(x) = 8000 + 160x\) represents the number of fish is a lake, where \(x\) is the number of months since January 1, 2012. The function \(g(x) = 6x^2 - 20x + 4000\) represents the volume of water in this lake in cubic meters, where \(x\) is still the number of months since January 1.

**Notation 8.8.** The notation \((\frac{f}{g})(x)\) means to divide the function \(f\) by the function \(g\). In other words:

\[
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}
\]

Write an equation for \((\frac{f}{g})(x)\).

What is \((\frac{f}{g})(12)\)? What are the units and what does this number represent?
Example 8.9. The following table shows the average SAT scores as a function of family income level in 2009. Let $x$ represent family annual income in thousands of dollars, $r(x)$ represent average reading score, $m(x)$ represent average math score, and $w(x)$ represent average writing score.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$r(x)$</th>
<th>$m(x)$</th>
<th>$w(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10$</td>
<td>434</td>
<td>457</td>
<td>430</td>
</tr>
<tr>
<td>$30$</td>
<td>462</td>
<td>475</td>
<td>453</td>
</tr>
<tr>
<td>$50$</td>
<td>488</td>
<td>497</td>
<td>478</td>
</tr>
<tr>
<td>$70$</td>
<td>503</td>
<td>512</td>
<td>491</td>
</tr>
<tr>
<td>$90$</td>
<td>517</td>
<td>528</td>
<td>505</td>
</tr>
<tr>
<td>$110$</td>
<td>525</td>
<td>538</td>
<td>516</td>
</tr>
<tr>
<td>$130$</td>
<td>529</td>
<td>542</td>
<td>520</td>
</tr>
<tr>
<td>$150$</td>
<td>536</td>
<td>550</td>
<td>527</td>
</tr>
<tr>
<td>$180$</td>
<td>542</td>
<td>554</td>
<td>535</td>
</tr>
</tbody>
</table>

Find

a) $(r + m)(90)$

b) $(m - w)(50)$

Example 8.10. The graphs of the functions $f(x)$ and $g(x)$ are drawn below.

Find

a) $(f + g)(1)$

b) $(f \circ g)(4)$

c) $(\frac{f}{g})(0)$
Example 8.11. Two children decided to run a lemonade and hot chocolate stand. The function

\[ f(x) = 3|x - 65| + 10 \]

represents the number of drinks sold as a function of temperature in degrees Fahrenheit. The function

\[ g(x) = 0.5 \times x - 5 \]

represents the profit in dollars as a function of the number of drinks sold.

(If time) What does the number 0.5 mean in this problem? (what are its units)
(If time) What does the number 5 mean in this problem?
(If time) What temperature is the worst for drink sales?

The kids want to know how much they’ll make if it is 75° degrees out. Find the answer for them.
**Definition 8.12.** For two functions \( f \) and \( g \), the composition of \( g \) and \( f \) is

\[
g \circ f(x) = g(f(x))
\]

This means, apply \( f \) to \( x \) first, and then apply \( g \) to the answer. In the drink stand example,

\[
g \circ f(75) = g(f(75)) = g(40) = 15
\]

**Note 8.13.** Be careful: \( g \circ f(x) \) is not the same thing as \( g \cdot f(x) \). The expression \( g \cdot f(x) \) means the same thing as \( (gf)(x) \), which is multiplication. \( g \circ f(x) \) means composition, which is different from multiplication.

**Example 8.14.** The tables below define the functions \( f \) and \( g \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>8</td>
<td>3</td>
<td>6</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>10</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Find:

a) \( g \circ f(3) \)

b) \( g \circ f(4) \)

c) \( f \circ g(4) \)

d) \( f \circ f(2) \)

e) \( f \circ g(6) \)

f) Challenge: what is the domain of \( f \)? the domain of \( g \)? the domain of \( f \circ g \)? the domain of \( g \circ f \)?
Example 8.15. The following graphs represent two functions $h$ and $p$.

Find

a) $h \circ p(0)$

b) $p \circ h(-3)$

c) $p \circ p(0)$

We’ve done composition problems with tables and graphs, and now it’s time to turn our attention to formulas.

Example 8.16. Let $g(x) = \sqrt{9 - x}$. Find:

a) $g(3)$

b) $g(b)$

c) $g(x^2)$

d) $g(5x + 2)$

Example 8.17. Let $f(x) = 5x + 2$. Let $g(x) = \sqrt{9 - x}$. Find $g \circ f(x)$. Hint: remember that $g \circ f(x)$ means $g(f(x))$, and work from the inside out.
Example 8.18. Let $p(x) = x^2 + x$. Let $q(x) = -2x$. Find:

a) $q \circ p(x)$

b) $p \circ q(x)$

c) $p \circ p(x)$

Example 8.19. $h(x) = \sqrt{x^2 + 7}$. Find functions $f$ and $g$ so that $h(x) = f \circ g(x)$.

Example 8.20. (If extra time.) $r(x) = (7x + 2)^3$. Find $f$ and $g$ such that $r(x) = f \circ g(x)$.

Example 8.21. (If extra time) Let $f(x) = \frac{1}{x}$ and $g(x) = 3x + 1$. Find

a) $f \circ g(x)$

b) $g \circ f(x)$

c) $f \circ f(x)$

d) $g \circ g(x)$

Note 8.22. In general, $f \circ g \neq g \circ f$!
Example 9.1. Suppose \( f(x) \) is the function defined by the chart below:

\[
\begin{array}{c|cccc}
  x & 2 & 3 & 4 & 5 \\
  f(x) & 3 & 5 & 6 & 1 \\
\end{array}
\]

In other words,

- \( f(2) = 3 \)
- \( f(3) = 5 \)
- \( f(4) = 6 \)
- \( f(5) = 1 \)

Draw the graph of \( y = f(x) \) below.

Definition 9.2. The inverse function for \( f \), written \( f^{-1}(x) \), undoes what \( f \) does.
Since $f$ takes 2 to 3, $f^{-1}$ takes 3 to 2. In other words, $f^{-1}(3) = 2$.

Fill in the blanks for the other values that $f$ and $f^{-1}$ trade back and forth, and then fill in the statements below.

\[
\begin{array}{c|c|c|c}
  x & f^{-1}(x) & 3 & 2 \\
\end{array}
\]

Notice how the $x$ and $y$ values are switched in the table for $f^{-1}$ compared to the original table for $f$.

**Note 9.3. Key Fact 1**: Inverse functions reverse the roles of $y$ and $x$. 

Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same axes below (using two different colors, or two different marks for the points). What do you notice about the points on the graph of $y = f(x)$ and the points on the graph of $y = f^{-1}$?

Note 9.4. Key Fact 2: The graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by reflecting over the line ________.

In our same example, compute:

- $f^{-1}(f(2)) = $  
- $f^{-1}(f(3)) = $  
- $f^{-1}(f(4)) = $  
- $f^{-1}(f(5)) = $  
- $f^{-1}(f(6)) = $  
- $f^{-1}(f(1)) = $  

Note 9.5. Key Fact 3: $f^{-1}(f(x)) = $ and $f(f^{-1}(x)) = $ . This is the mathematical way of saying that $f$ and $f^{-1}$ undo each other.

Example 9.6. $f(x) = x^3$. Guess what the inverse of $f$ should be. Remember, $f^{-1}$ undoes the work that $f$ does.

Example 9.7. $h(x) = 3x + 1$. Guess what the inverse of $h$ should be. How can you check to see if your guess is correct? Test your guess!
Example 9.8. If \( f(x) = \frac{x}{2} - 1 \), which of these is \( f^{-1}(x) \)? Prove that your answer is correct!

a) \( g(x) = 2x + 1 \)
b) \( h(x) = 2x + 2 \)
c) \( p(x) = \frac{2}{x} + 1 \)

Example 9.9. \( h(x) = 7 - x^3 \). Find \( h^{-1}(x) \) by reversing the roles of \( y \) and \( x \) and solving for \( y \).

Example 9.10. If you are age \( x \), the oldest person that it is okay for you to date is given by the formula \( d(x) = 2x - 14 \). Plug in your own age for \( x \) and see how old a person you can date.

Suppose you want to date a younger person instead of an older person. Invert the formula to find out how young a person someone of a given age can date.

Plug in your own age for \( x \) into \( d^{-1}(x) \) and see how young a person you can date.
Example 9.11. Find the inverse of the function:

\[ f(x) = \frac{5 - x}{3x} \]

Note 9.12. \( f^{-1}(x) \) means the inverse function for \( f(x) \). Note that \( f^{-1}(x) \neq \frac{1}{f(x)} \).

Example 9.13. Find the inverse of the function:

\[ f(x) = -\frac{7}{x} + 1 \]

Question 9.14. Do all functions have inverse functions? That is, for any function that you might encounter, is there always a function that is its inverse?

Try to find an example of a function that does not have an inverse function.
Question 9.15. The graphs of graphs four functions are drawn below. Which of these functions have inverse functions?

Note 9.16. Key Fact 4: A function $f$ has an inverse function if and only if the graph of $f$ satisfies the horizontal line test (i.e. every horizontal line intersects the graph of $y = f(x)$ in at most one point.)

Why?

Definition 9.17. A function is one-to-one if it passes the horizontal line test. Equivalently, a function is one-to-one if for any two different $x$-values $x_1$ and $x_2$, $f(x_1)$ and $f(x_2)$ are different numbers. Sometimes, this is said: $f$ is one-to-one if, whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$. 

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Example 9.18. Which of these function are one-to-one?

\[ \ell(x) = \frac{5}{x} \]
\[ m(x) = \frac{1}{x} \]
\[ n(x) = 0.5x^3 - 4.5x \]
\[ p(x) = \sqrt{x - 2} \]

Example 9.19. (Tricky) Find \( p^{-1}(x) \), where \( p(x) = \sqrt{x - 2} \) drawn above. Graph \( p^{-1}(x) \) on the same axes as \( p(x) \) above.

For the previous function \( p(x) = \sqrt{x - 2} \), what is:
- the domain of \( p \)?
- the range of \( p \)?
- the domain of \( p^{-1} \)?
- the range of \( p^{-1} \)?

Note 9.20. Key Fact 5: For any invertible function \( f \), the domain of \( f^{-1}(x) \) is ________________, and the range of \( f^{-1}(x) \) is ________________.
11 Class 11 - §5.2 - Exponential Functions

Definition 11.1. Book’s definition: An exponential function is a function that can be written in the form
\[ f(x) = a^x \]
where \( a \) is a real number and \( a > 0 \).

Definition 11.2. More general and more standard definition: An exponential function is a function that can be written in the form
\[ f(x) = k \cdot a^x \]
where \( a \) and \( k \) are real numbers, \( k \neq 0 \), and \( a > 0 \).

Question 11.3. Why do we require that \( k \neq 0 \)?

Question 11.4. Why do we require that \( a > 0 \)?

Example 11.5. Which of these are exponential functions?
1. \( f(x) = 3^x \)
2. \( g(x) = 0.2^x \)
3. \( h(x) = x^8 \)
4. \( \ell(x) = \pi^x \)
5. \( n(x) = 2^{-x} \)
6. \( m(x) = \left( \frac{1}{3} \right)^x \)

Example 11.6. Match the graph to the function.
a) \( y = 2^x \)
b) \( y = 3^x \)
c) \( y = 1.1^x \)
d) \( y = 0.8^x \)
e) \( y = 0.5^x \)
Question 11.7. How can you tell from the equation $y = a^x$ whether the function is increasing or decreasing?

Note 11.8. For $f(x) = a^x$, if __________, then $f(x)$ is an increasing function. If __________, then $f(x)$ is a decreasing function.

Note 11.9. If $a$ is close to 1, then the graph of the function $f(x) = a^x$ looks __________. If $a$ is far from 1, then the graph looks __________.

Question 11.10. What does the graph of $f(x) = a^x$ look like if $a = 1$?

Question 11.11. What does the graph of $f(x) = a^x$ look like if $a < 0$?

Question 11.12. What is the domain and range of $f(x) = 5^x$?

Question 11.13. What happens to the graph of $f(x) = 5^x$ as $x \to \infty$? (Note: as $x \to \infty$ is the mathematical way of saying “as $x$ gets very large”.)

Question 11.14. What happens to the graph of $f(x) = 5^x$ as $x \to -\infty$? (Note: $x \to -\infty$ is the mathematical way of saying that $x$ gets more and more negative, passing through values like -100, -1000, -10,000, etc.)

Question 11.15. What happens to the graph of $f(x) = 0.3^x$ as $x \to \infty$?
Question 11.16. What happens to the graph of \( f(x) = 0.3^x \) as \( x \to -\infty \)?

Definition 11.17. A horizontal asymptote is a horizontal line that a graph approaches as \( x \to \infty \) or as \( x \to -\infty \).

Question 11.18. What is the horizontal asymptote for the function \( y = 1.3^x \)?

What is the horizontal asymptote for the function \( y = 2 \cdot 1.3^x + 5 \)?

Note 11.19. The most famous exponential function in the world is \( f(x) = e^x \). This function is sometimes written as \( f(x) = \exp(x) \). The number \( e \) is Euler’s number, and is approximately 2.781828172853...

Example 11.20. (If time) Which function do you want to represent your annual income in dollars at year \( x \)?

- \( y = 2^x \)
- \( y = x^8 \)

Why? Which will hit \$1,000,000 first?
12 Class 12 - §5.3 - Logarithms

Definition 12.1. \( \log_a b = c \) means \( a^c = b \).

Note 12.2. You can think of logarithms as exponents: \( \log_a b \) is the exponent (or “power”) that you have to raise \( a \) to, in order to get \( b \). The number \( a \) (on the left side of the equation above) is called the base of the logarithm. Notice that on the right side of the equation, \( a \) is also the base of the exponent.

Example 12.3.
\[
\log_2 8 = 3 \text{ because } 2^3 = 8 \\
\log_2 y = \square \text{ means } 2^\square = y
\]

Example 12.4. Evaluate the following expressions by hand by rewriting them using exponents instead of logs:

a) \( \log_2 32 = \Box \)
b) \( \log_2 2 = \Box \)
c) \( \log_2 \frac{1}{2} = \Box \)
d) \( \log_2 \frac{1}{3} = \Box \)
e) \( \log_2 1 = \Box \)

Example 12.5. Evaluate the following expressions by hand by rewriting them using exponents instead of logs:

a) \( \log_{10} 1,000,000 = \Box \)
b) \( \log_{10} \sqrt{100} = \Box \)
c) \( \log_{10} 0 = \Box \)
d) \( \log_{10} -100 = \Box \)

Notation 12.6. \( \ln x \) means \( \log_e x \), and is called the natural log.

\( \log x \), with no base, means \( \log_{10} x \) and is called the common log.

You can find \( \ln x \) and \( \log x \) for various values of \( x \) using the buttons on your calculator.

Example 12.7. Rewrite the following using logs. Do not solve for any variables.

a) \( e^2 = 7.389 \)
b) \( 10^{-5} = 0.0001 \)
c) \( e^{3x+7} = 4 - y \)


a) \( \log 13 = 1.11394 \)
b) \( \ln \frac{1}{2} = -1 \)
c) \( \log(z + 3) = 2w \)

**Example 12.9.** (If time) Solve for the variable by rewriting the log as an exponent.

a) \( \log_3(2y) = 4 \)

b) \( \log_{x+3} 0.01 = -2 \)

**Example 12.10.** Fill in the following table of values and use it to draw a graph of \( y = \log_2 x \).

\[
\begin{array}{c|c|c|c|c|c|c}
 x & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 & 8 \\
 y = \log_2 x & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

**Example 12.11.** Fill in the following table of values and use it to draw a graph of \( y = 2^x \) on the same axes above.

\[
\begin{array}{c|c|c|c|c|c|c}
 x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
 y = 2^x & 6 & 2 & 0.5 & 1 & 2 & 4 & 8 \\
\end{array}
\]

**Question 12.12.** How are the functions \( f(x) = \log_2 x \) and \( g(x) = 2^x \) related?

**Example 12.13.** Compute the following, where \( f(x) = \log_2 x \) and \( g(x) = 2^x \). Write out each step.

\[
f \circ g(x) \quad g \circ f(x)
\]
Note 12.14. Based on the computations above, \( \log_2(2^x) = \) \( \) and \( 2^{\log_2(x)} = \) \( \).

Note 12.15. For any base \( a \), \( \log_a(a^x) = \) \( \) and \( a^{\log_a(x)} = \) \( \).

Example 12.16. Find
a) \( \log_5 5^3 \)

b) \( 3^{\log_3 1.4} \)

c) \( \ln(e^x) \)

d) \( 10^{\log(3z)} \)

Notation 12.17. The convention is that exponentiation and multiplication come first, then taking the log, then addition. So

\( \log_5 5^3 \) means the same thing as \( \log_5(5^3) \) (exponentiation is done first)

\( \log 3z \) means \( \log(3z) \)

but \( \log 5 + w \) means \( (\log(5)) + w \), and if you want to do addition first instead, then you have to put it in parentheses like \( \log(5 + w) \)

It is best to be generous with parentheses anyway to avoid confusion!

Question 12.18. (If time) For what kinds of values does a logarithm produce a positive value? For what kinds of values does a logarithm produce a negative value? From an algebraic standpoint, why does this occur?

Question 12.19. (If time) For what value is \( \log x = 1 \)?

Example 12.20. (If time) Some values of \( \log x \) are easy to compute by hand and others are harder, but can still be estimated by hand. Explain why \( 2 < \log 461 < 3 \).

Example 12.21. (If time) Compute the following on your calculator. What do you notice?

a) \( \log 16 = \) \( \)

b) \( \log 1.6 = \) \( \)

\( \log 1600 = \) \( \)

d) \( \log 3 = \) \( \)

e) \( \log 0.3 = \) \( \)

f) \( \log 30 = \) \( \)

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Question 12.22. What is the relationship between $\log a$ and $\log(a \cdot 10)$? What is the relationship between $\log a$ and $\log a \cdot 10^n$ for any $n$?
13 Class 13 - §5.3 Continued - Log Rules

Note 13.1. Let's review some exponent rules:

1. $2^0 =$
2. $2^n \cdot 2^m =$
3. $\frac{2^n}{2^m} =$
4. $(2^n)^m =$

The next few exercises will help you figure out how to compute similar rules for logs.

Example 13.2. What is log 1? What is ln 1? What is log_7 1? What is log_a 1? Why?

Example 13.3. (If time)

1. Find log 36.
4. Note that $4 \times 9 = 36$. Find log 4 and log 9.
5. Note that $3 \times 12 = 36$. Find log 3 and log 12.

Question 13.4. (If time) What relationships to you notice between log(36) and log of the numbers related to 36?

Note 13.5. The product rule for logarithms says that log(xy) =

In fact, the product rule is true for a log with any base, not just the base of 10. That is,

$log_a(xy) =$

In your own words, state the product rule.
Proof. of the product rule for base 10. (Optional) For each line in the proof, provide a reason to justify that line.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( r = \log_{10} x ) and ( s = \log_{10} y ), then ( x = 10^r ) and ( y = 10^s ).</td>
<td></td>
</tr>
<tr>
<td>So ( x \cdot y = 10^r \cdot 10^s ).</td>
<td></td>
</tr>
<tr>
<td>So ( x \cdot y = 10^{r+s} ).</td>
<td></td>
</tr>
<tr>
<td>Therefore, ( \log_{10}(x \cdot y) = r + s ).</td>
<td></td>
</tr>
<tr>
<td>So ( \log_{10}(x \cdot y) = \log_{10} x + \log_{10} y ).</td>
<td></td>
</tr>
</tbody>
</table>

**Question 13.6.** Which exponent rule is the product rule related to?

We’ve found a way to re-express the log of a product. Make a conjecture of how to re-express the log of a quotient:

**Note 13.7.** The **quotient rule** for logarithms says that \( \log_{b}(\frac{x}{y}) = \). 

In your own words, state the quotient rule.

Proof. of the quotient rule for base 10. (Optional) Write down a proof of the quotient rule along the lines of the previous proof of the product rule.

**Question 13.8.** What exponent rule is the quotient rule related to?
There is another log rule having to do with exponents. Make a conjecture how to express $\log(6^2)$ and $\log(6^3)$ in terms of $\log 6$. (Hint: use the product rule.)

$$\log(6^2) = \underline{\quad}$$

$$\log(6^3) = \underline{\quad}$$

**Note 13.9.** The **power rule** says that $\log_b(x^m) = \underline{\quad}$. 

In your own words, state the power rule.

**Proof.** of the power rule for base 10. (Optional) For each line in the proof, provide a reason to justify that line.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $r = \log_{10} x$, then $x = 10^r$.</td>
<td></td>
</tr>
<tr>
<td>So $x^m = (10^r)^m$.</td>
<td></td>
</tr>
<tr>
<td>So $x^m = 10^{rm}$.</td>
<td></td>
</tr>
<tr>
<td>So $\log_{10} x^m = r \cdot m$.</td>
<td></td>
</tr>
<tr>
<td>Therefore, $\log_{10} x^m = m \log_{10} x$.</td>
<td></td>
</tr>
</tbody>
</table>

As review, please fill out the chart of the exponent rules and the related log rules.

<table>
<thead>
<tr>
<th>Exponent Rule</th>
<th>Log Rule</th>
<th>Name of Log Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^0 = 1$</td>
<td></td>
<td>–</td>
</tr>
<tr>
<td>$a^n \cdot a^m = a^{n+m}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{a^m}{a^n} = a^{n-m}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(a^n)^m = a^{nm}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 13.10.** Rewrite the following as a sum or difference of logs:

a) $\log \left( \frac{x}{y} \right)$

b) $\ln \left( \frac{e^{\pi i}}{z} \right)$

c) $\log(5 \cdot 2^i)$
Example 13.11. Rewrite as a single log:

a) \( \log_5 a - \log_5 b + \log_5 c \)

b) \( 4 \ln z - 5 \ln w \)

c) \( \log(4t) - 3 \log(2t) \)

d) \( \ln(x + 1) + \ln(x - 1) - 2 \ln(x^2 - 1) \)
Example 14.1. Solve: $5^x - 2 = 9$

Example 14.2. Solve: $2 \cdot 3^{5x - 1} - 5 = 7$

Example 14.3. Solve: $\ln(2x + 5) = 0.1$
Example 14.4. Solve: \( \log(x + 5) - \log(3x) = 1 \)

Example 14.5. Solve: \( 4^{x+1} = 6^{2-3x} \)

Example 14.6. (If time) Solve: \( 100e^{-0.4x} = 200e^{-0.6x} \)
Section 5.4 starts here.

We saw previously that the graphs of \( y = \log_2(x) \) and \( y = 2^x \) are reflections of each other over the line \( y = x \), because they are inverse functions.

**Example 14.7.** Label the two functions in the graph below:

![Graph of functions](image)

One the two functions has a horizontal asymptote and one has a vertical asymptote. Draw the asymptotes on the graph as dotted lines. An asymptote is a horizontal or vertical line that a graph approaches.

**Example 14.8.** Without your calculator, match these equations to the functions on the graph below:

- a) \( y = \log x \)
- b) \( y = \ln x \)
- c) \( y = 10^x \)
- d) \( y = e^x \)
Fill in the table for the two functions:

<table>
<thead>
<tr>
<th></th>
<th>$y = 10^x$</th>
<th>$y = \log x$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Domain</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Range</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Asymptote</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 14.9.** Find the domain of $y = \log(x - 4) + 7$.

**Example 14.10.** Find the inverse for $y = \log(x - 4) + 7$.

**Example 14.11.** Find the inverse of $f(x) = 42 \cdot e^{x^2}$. 

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It is easy to compute log base 10 and log base e on your calculator. There is a formula for computing logs with other bases in terms of log\(_{10}\) or log\(_{e}\), which you will discover in the following example.

**Example 14.12.** Find \( \log_5 17 \), in terms of \( \log_{10} \) using the following steps:
1) Set \( x = \log_5 17 \). We want to find \( x \).
2) Rewrite this equation as an exponential equation.
3) Take the log base 10 of both sides.
4) Use the power rule to rewrite.
5) Solve for \( x \).

To summarize \( \log_5 17 = \frac{\log 17}{\log 5} \)

**Note 14.13.** Change of Base Formula: In general,

\[
\log_a(b) = \frac{\log b}{\log a}
\]

True or False review of log rules:
1) \( \log(3x) = (\log 3)(\log x) \)
2) \( \log \frac{1}{a} = \frac{1}{\log a} \)
3) \( \log(2^3 \cdot 7^2) = 3\log 2 + 2\log 7 \)
4) \( \log 0 = 1 \)
5) \( \log(10^{-2} + 10^3) = -2 + 3 = 1 \)
6) \( \ln(10^3) = 3 \)
7) \( \log(5 - z) = \frac{\log 5}{\log z} \)
15   Class 15 - §3.1 - Complex Numbers

There is no real number that is equal to $\sqrt{-1}$. (Why?) But we can invent a name for $\sqrt{-1}$ and work with it anyway.

**Definition 15.1.** The letter $i$ represents $\sqrt{-1}$

**Question 15.2.** What is $i^2$?

**Example 15.3.** Once we have $i$, we can figure out the square root of other negative numbers. For example, $\sqrt{-9} = \sqrt{-1} \cdot \sqrt{9} = i \cdot 3 = 3i$

**Example 15.4.** a) Find $\sqrt{-25}$.

b) Find $\sqrt{-75}$

**Definition 15.5.** An imaginary number is a number that can be written in the form $b \cdot i$, where $b$ is a real number, and $b \neq 0$.

For example, $3i$, $\sqrt{2}i$, $i \cdot \pi$, and $\sqrt{-25}$ are all imaginary numbers.

**Example 15.6.** Add: $4i + 2i =$

**Note 15.7.** When you add two imaginary numbers, you get another imaginary number.

**Example 15.8.** Add: $5 + 2i =$

**Note 15.9.** When you add a real number and an imaginary number, you can’t combine terms. You get a new kind of number, called a complex number.

**Definition 15.10.** A complex number is a number that can be written in the form $a + bi$ where $a$ and $b$ are real numbers, and $a$ and/or $b$ CAN be zero. The number $a$ is called the real part and $bi$ is called the imaginary part of the complex number. Some people (including the authors of the book) call $b$ the imaginary part, instead of $bi$.

**Example 15.11.** For each number, find the real part, the imaginary part, and determine whether the number is real, imaginary, and/or complex.

<table>
<thead>
<tr>
<th>Number</th>
<th>Real Part</th>
<th>Imaginary Part</th>
<th>Is it real?</th>
<th>Is it imaginary?</th>
<th>Is it complex?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$77i + 4.2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$5 - \frac{3}{2}i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-7i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Squares and rectangles are related: every square is also a rectangle but not every rectangle is also a square. Decide which of the following statements about real number, imaginary numbers, and complex numbers are true.

**Question 15.12.** Is every real number also a complex number?

**Question 15.13.** Is every imaginary number also a complex number?

**Question 15.14.** Is every complex number either a real number or an imaginary number?

**Example 15.15.** Which Venn diagram describes the relationship between real, imaginary, and complex numbers?

![Venn diagrams](image1)

**Example 15.16.** If \( x = 3 + 4i \) and \( y = 7 - 2i \), find \( x - y \) and simplify.

**Example 15.17.** If \( x = 3 + 4i \) and \( y = 7 - 2i \), find \( x \cdot y \) and simplify.

**Example 15.18.** For \( r = 2 - 3i \) and \( s = 7 + 6i \), find \( r + s, r - s \) and \( r \cdot s \) and simplify.
We can add, subtract, and multiply complex numbers and the result is always a complex number. Dividing two complex numbers is more complicated and involves something called the complex conjugate.

**Definition 15.19.** The complex conjugate of $a + bi$ is $a - bi$.

**Example 15.20.** Find the complex conjugates:

a) $7 + 3i$
b) $2 - 4i$
c) $-9.2i$
d) $3$

**Example 15.21.** $x = -2 + 3i$. Find the conjugate of $x$ and multiply $x$ by its conjugate.

**Example 15.22.** $x = 3 - 7i$. Find the conjugate of $x$ and multiply $x$ by its conjugate.

**Note 15.23.** The product of a complex number and its conjugate is always a _______ number.

**Proof.**

$$(a + bi)(a - bi) =$$
Example 15.24. Find
\[
\frac{6 + 5i}{3 + i}
\]
Trick: multiply the numerator and the denominator by the conjugate of the denominator.

Note 15.25. Your calculator may be able to divide complex numbers correctly, but you should still know how to do it yourself.

Example 15.26. Find
\[
\frac{7 + 2i}{-2i}
\]
Example 15.27. Simplify the powers of \( i \), and look for a pattern:

\[
i = i \\
i^2 = -1 \\
i^3 = i^2 \cdot i = -1 \cdot i = -i \\
\]

\[
i^4 = \\
i^5 = \\
i^6 = \\
i^7 = \\
i^8 = \\
i^9 = \\
i^{10} = \\
i^{11} = \\
i^{12} = 
\]

Example 15.28. What is \( i^{27} \)?

Example 15.29. Find:

a) \( i^{81} \)

b) \( i^{-9} \)
16 Class 16 - §3.2 - Quadratic Equations

Definition 16.1. Quadratic functions are functions that can be written in the form

\[ f(x) = ax^2 + bx + c \]

where \( a \neq 0 \)

Example 16.2. Which of these are quadratic functions?

I) \( f(x) = 12 - 30x - 3x^2 \)

II) \( f(x) = x^2 - 5.7 \)

III) \( f(x) = 3x + 1 \)

IV) \( f(x) = 2(x - 1)^2 + 3 \)

Example 16.3. Use your knowledge of transformations of functions to sketch the graph the following quadratic functions. Use your calculator to check your answer. The shape of these graphs is called a parabola. Draw a dotted vertical line on each graph to show the line of mirror symmetry (called the axis of symmetry). Label the coordinates for the vertex, which is where the graph has a maximum or minimum.

1. \( f(x) = 2(x - 1)^2 + 3 \) (this is the same example IV as above)

2. \( y = (x + 2)^2 + 5 \)

3. \( y = -2(x + 4)^2 - 3 \)
Definition 16.4. A quadratic function written in the form \( y = a(x - h)^2 + k \) is said to be in vertex form. A quadratic function written in the form \( y = ax^2 + bx + c \) is said to be in standard form.

Note 16.5. A function in the form \( y = a(x - h)^2 + k \) has a vertex at \((h, k)\). It opens up (happy face) if \(a > 0\) and opens down (frowny face) if \(a < 0\).

Example 16.6. What is the vertex of the following functions? Which ones are happy faces and which are frowny faces?

1. \( y = 4(x - 5.2)^2 + 6.7 \)
2. \( y = \frac{7}{2}(x + \frac{8}{3})^2 + \frac{1}{2} \)
3. \( y = -3(x + 1.4)^2 - 2.2 \)

When you are given a quadratic function in vertex form, it is easy to find the vertex. When it is given in standard form, instead, it is harder to find the vertex. One way to find it is to rewrite the function in vertex form using a technique called completing the square.

Example 16.7. Complete the square to put the function \( f(x) = -3x^2 - 30x + 12 \) in vertex form, using the following steps.

Step 1: Divide both sides by the coefficient of \(x^2\).

Step 2: Add or subtract terms so that the terms involving \(x\) are all on the right and the terms that don’t involve \(x\) are on the left.

Step 3: Find a number to add to both sides that will make the right side a perfect square. Hint: take the coefficient of \(x\), divide it by 2, and square it. Why does this work?

Step 4: Rewrite the right side as a perfect square (that is, in the form \((x - h)^2\), or in the form \((x + h)^2\) for some number \(h\)).

Step 5: Solve for \(y\).
Example 16.8. For each expression, what number do you have to add to make a perfect square?

1. $x^2 + 10x$
2. $x^2 + 6x$
3. $x^2 - 6x$
4. $x^2 + 5x$
5. $x^2 + nx$

Example 16.9. Complete the square and find the vertex:

\[ y = 2x^2 - 28x + 6 \]

Example 16.10. Complete the square and find the vertex:

\[ y = x^2 - 3x + 7 \]
Completing the square is a lot of work, so fortunately, there is a shortcut formula. Based on the following table of examples, see if you can guess how to find the x-coordinate of the vertex just from the coefficients $a$, $b$, and $c$ in the standard form.

<table>
<thead>
<tr>
<th>Equation in standard form</th>
<th>x-coord of vertex ($h$)</th>
<th>y-coord of vertex ($k$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 10x + 7$</td>
<td>−5</td>
<td>−18</td>
</tr>
<tr>
<td>$x^2 + 2x + 5$</td>
<td>−1</td>
<td>4</td>
</tr>
<tr>
<td>$x^2 - 3x + 7$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{19}{4}$</td>
</tr>
<tr>
<td>$-3x^2 - 30x + 12$</td>
<td>−5</td>
<td>87</td>
</tr>
<tr>
<td>$2x^2 - 28x + 6$</td>
<td>7</td>
<td>−92</td>
</tr>
</tbody>
</table>

**Note 16.11.** In general, the quadratic function $p(x) = ax^2 + bx + c$ can be written in the vertex form $y = a(x - h)^2 + k$, where $h =$ _______ and $k =$ _______. This is called the **vertex formula**.

**Proof.** (optional) You can prove the vertex formula by completing the square in general from the standard form $p(x) = ax^2 + bx + c$.

**Example 16.12.** Use the vertex formula to find the vertex and rewrite the equation in vertex form:

$$y = 5x^2 + 20x + 2$$

**Example 16.13.** Use the vertex formula to find the vertex and rewrite the equation in vertex form:

$$y = 3x^2 - 6x - 4$$
17  Class 17 / 18 - §3.3 - Quadratic Equations and Inequalitites

Definition 17.1. A x-intercept of a function $f(x)$ is ________________.

Definition 17.2. A zero of a function $f(x)$ is ________________.

Example 17.3. Find the x-intercepts of the graph of $y = 2x^2 + 6x - 56$.

Example 17.4. Find the zeros of the function $p(x) = 2x^2 + 6x - 56$.

Example 17.5. Find the solutions to the equation $2x^2 + 5x - 56 = 0$.

Note 17.6. All of these problems are equivalent. All of them can be solved in any of three ways:

1. __________
2. __________
3. __________
Note 17.7. The solutions to the equation
\[ ax^2 + bx + c = 0 \]
are
\[ x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a} \]
This is called the \textbf{quadratic formula}.

\textbf{Example 17.8}. Find the x-intercepts of \( p(x) = x^2 - x - 1 \).

\textbf{Example 17.9}. Solve \( 9x^2 + 8 = 12x \)
Example 17.10. Find the zeros of $f(x) = -x^2 + 10x - 25$

Example 17.11. Graph the three quadratic functions in the previous three examples.
Definition 17.12. When using the quadratic equation, the part of the equation under the square root sign: $b^2 - 4ac$ is called the discriminant and plays an important role.

Fill in the following table.

<table>
<thead>
<tr>
<th>discriminant $b^2 - 4ac$</th>
<th>number and type of solutions (real or not real)</th>
<th>number of intercepts</th>
<th>typical graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^2 - 4ac &gt; 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b^2 - 4ac &lt; 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b^2 - 4ac = 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 17.13. Use the discriminant to determine the number of x-intercepts of the graph of $g(x) = 5x^2 - 3x + 1$.

Example 17.14. (if time) Use the discriminant to determine the number of solutions of the equation $x^2 + 9 = 6x$.

Proof. (optional) The quadratic formula can be proved by completing the square, using the following steps:

Step 1: Divide all terms by $a$.
Step 2: Move all terms that don’t involve $x$ to the left side.
Step 3: Add the same thing to both sides to complete the square on the right side.
Step 4: Take the square root of both sides.
Step 5: Solve for $x$. 

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Solving quadratic inequalities is a bit more complicated than solving quadratic equations. It can be done using algebra or using a graph.

**Example 17.15.** Solve $3x - 9 < 0$. (This is just a linear inequality.)

**Example 17.16.** Solve $x^2 - 4 > 0$.

To solve a quadratic inequality correctly, one algebraic method is to first solve the related quadratic equation and put the values on a number line. Then test values of $x$ between those solutions to see where the expression is positive and negative.

Solve $x^2 - 4 = 0$ and plot the solutions on a number line.

Put a + sign in the gap between the solutions if $x^2 - 4$ is positive there, and a - sign if $x^2 - 4$ is negative.

Use the +’s and -’s to decide where $x^2 - 4 > 0$.

**Example 17.17.** Solve $x^2 + 7x - 8 < 0$ algebraically (using the number line method).
Example 17.18. Graph \( f(x) = x^2 + 7x - 8 \) and circle the part of the graph below the x-axis. What is the relationship between the circled part of the graph and the answer to the previous problem?

Example 17.19. Solve \( 4x \leq x^2 + 2 \)

Example 17.20. Solve \( 2x^2 - x + 4 \geq 0 \).
Example 17.21. Graph \( f(x) = 2x^2 - x + 4 \) and circle the part of the graph above the x-axis. What is the relationship between the circled part of the graph and the answer to the previous problem?

Example 17.22. (if time) Use the graph of \( y = f(x) \) below to solve the equation \( f(x) < 0 \).
18 Class 19 and 20 - §3.5 - Polynomials

Definition 18.1. A polynomial is a function that can be written in the form

\[ p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 , \]

where \( a_n, a_{n-1}, \ldots, a_1, a_0 \) are real numbers and \( n \) is a positive integer.

Which of the following are polynomials?

1. \( f(x) = 7.5x^6 - 32x^3 + 5x^2 - 4.1x + 6 \)
2. \( g(x) = 6x - \frac{3}{2}x^2 + 17 \)
3. \( q(x) = 1.2x^{-1} + 3x + 7 \)
4. \( h(x) = 4x - 7 \)
5. \( r(x) = 4\sqrt{x} + x^3 \)

Definition 18.2. The degree of the polynomial is the largest exponent. The leading term is the term with the largest exponent, and the leading coefficient is the number in the leading term. A constant term is a term with no \( x \)'s in it.

Example 18.3. For \( p(x) = 5x^3 - 3x^2 - 7x^4 + 2x + 18 \), what is the

- degree?
- leading term?
- leading coefficient?
- constant term?

Definition 18.4. In the graph below, the marked points are called turning points. They are also called extreme points. The are also called maximum and minimum points.

Definition 18.5. A global (or absolute) maximum point is the highest point anywhere. A local (or relative) maximum point is the highest point in the neighborhood.
Definition 18.6. A **global (or absolute) minimum point** is the lowest point anywhere. A **local (or relative) minimum point** is the lowest point in the neighborhood.

Example 18.7. Mark the global and local maximum and minimum points on the graph below.

![Graph](image)

**Note 18.8.** Maximum or minimum **points** means the \((x, y)\) coordinate. Maximum or minimum **values** means just the \(y\)-value.

Example 18.9. Graph \(f(x) = x^4 + 2x^3 - 15x^2 - 12x + 36\) on your calculator.

To get a good viewing window, you can start with ZOOM \(\rightarrow\) FIT, which gives the large scale view but obscures the details. Then hit the WINDOW button to manually decrease the values of YMIN and YMAX until you get a reasonable picture.

Use CALC \(\rightarrow\) MIN to find a local or global minimum. Use CALC \(\rightarrow\) MAX to find a local or global maximum. CALC \(\rightarrow\) ZERO to find one of the x-intercepts.
Definition 18.10. The **end behavior** of a function is how the “ends” of the function look as $x \to \infty$ and $x \to -\infty$.

You can draw the end behavior with little arrows.

**Example 18.11.** Determine the end behavior for these polynomials.

<table>
<thead>
<tr>
<th>polynomial</th>
<th>end behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = x^3 - 2x^2 + x - 2$</td>
<td></td>
</tr>
<tr>
<td>$g(x) = 5x^5 + 14x^3 + 7x$</td>
<td></td>
</tr>
<tr>
<td>$h(x) = -2x^7 - 4x^2 + 2$</td>
<td></td>
</tr>
<tr>
<td>$k(x) = 3x^2 + 2x - 1$</td>
<td></td>
</tr>
<tr>
<td>$\ell(x) = 5x^4 - 3x^3 + 2x^2$</td>
<td></td>
</tr>
<tr>
<td>$m(x) = -4x^2 + 7$</td>
<td></td>
</tr>
</tbody>
</table>

**Question 18.12.** Use the examples above to find a pattern. How can you tell from a polynomial’s equation what its end behavior will be? Hint: look at the leading term.

**Example 18.13.** What can you tell about the equation for the polynomial graphed below?
21 Class 21 and 22 - §3.6 and 3.7 - Theory of Polynomials

Example 21.1. Remember long division of numbers?

\[
7) 9371
\]

Example 21.2. Divide

\[
\frac{6x^3 - 7x^2 + 5}{2x - 1}
\]

Hint: to keep things lined up, it is handy to write the dividend as \(6x^3 - 7x^2 + 0x + 5\).
Example 21.3. Divide. Hint: write $x^2 + 2$ as $x^2 + 0x + 2$ to keep things lined up.

\[
\begin{array}{c}
4x^3 - 3x^2 + 2x + 7 \\
\hline
x^2 + 2
\end{array}
\]

Example 21.4. Let $f(x) = 3x^3 - 4x^2 - 27x + 10$. Find $f(x) \div (x - 2)$
Example 21.5. In the previous problem, what is $f(2)$?

Question 21.6. How is $f(2)$ related to the remainder when you divide $f(x)$ by $x - 2$? Why does this relationship hold?

Theorem 21.7. (The Remainder Theorem) If a polynomial $p(x)$ is divided by $x - k$, then the remainder is a number equal to $p(k)$.

Example 21.8. What is the remainder when you divide $g(x) = 3x^3 - 4x^2 + 5x - 12$ by $x - 1$?

Example 21.9. What is the remainder of $h(x) = x^2 - 7x - 8$ divided by $x + 1$?

Note 21.10. When we divide 66 by 11, we get a remainder of __________. Therefore, we know that 11 is a __________ of 66.

Similarly, when we divide $h(x) = x^2 - 7x - 8$ by $x + 1$, we get a remainder of __________. Therefore, we know that $x + 1$ is a __________ of $h(x) = x^2 - 7x - 8$.

Factor $h(x) = x^2 - 7x - 8$: 
Theorem 21.11. (The Factor Theorem)

\[(x - k)\text{ is a factor of the polynomial } p(x)\]

\[\iff \text{the remainder of } p(x) \text{ divided by } (x - k) \text{ is } 0.\]

\[\iff p(k) = 0.\]

Example 21.12. Is \(x - 3\) a factor of \(p(x) = 2x^3 - 3x^2 - 6x + 9\)?

Example 21.13. Suppose 2 and 3 are zeros of the quadratic function \(q(x)\). Find a possible equation for \(q(x)\). Hint: since 2 and 3 are zeros, this means \(q(2) = 0\) and \(q(3) = 0\). What does the factor theorem tell you about the factors of \(q(x)\)?

(challenge) What is the equation for \(q(x)\) if we also require that \(q(1) = 12\)?

Example 21.14. Find all real and complex zeros of \(f(x) = x^3 - 4x^2 + 3\) given that \(x = 1\) is a zero. Hint: use the factor theorem to find a factor of \(f(x)\), then factor \(f(x)\).
Example 21.15. Find all real and complex zeros of \( g(x) = x^3 + x^2 - 4x + 6 \), given that \( x = -3 \) is a zero.

Example 21.16. (optional) Factor \( p(x) = 2x^3 + 5x^2 - x - 6 \), given that \( x = -2 \) is a zero.
Example 21.17. (challenge) Find all zeros of \( q(x) = x^3 - 4x^2 + x + 6 \).

Example 21.18. (challenge) The polynomial \( h(x) \) is graphed below.

1. What are the real zeros of \( h(x) \)?
2. What is the remainder when \( h(x) \) is divided by
   
   (a) \( x - 6 \)?
   
   (b) \( x + 1 \)?
   
   (c) \( x + 3 \)?

3. What is the equation of \( h(x) \)?
Theorem 21.19. For a polynomial $P(x)$ and a complex number $k$ (that might be a real number), the following statements are equivalent:

a) $k$ is a zero of $P(x)$

b) 

c) 

d) 

e) For a real number $k$, the following is also equivalent:

Note 21.20. If $4 + 2i$ is a zero of quadratic function, then ______ is also a zero. Non-real complex zeros come in conjugate pairs.

Theorem 21.21. Conjugate Zeros Theorem Suppose $P(x)$ is a polynomial with real coefficients. If $a + bi$ is a zero of $P(x)$, then so is ______.

Example 21.22. a) Find a cubic polynomial that has a zero of 5 and also a zero of $2 - 3i$.

b) (challenge) Find such a polynomial with a y-intercept of 5.
Example 21.23. For the polynomial \( P(x) = x(x + 5)^2(x - 1)^3 \), what are the zeros of \( P(x) \)?

Definition 21.24. For a polynomial \( P(x) \), if \((x - k)^m\) is a factor of \( P(x) \), then \( k \) is called a zero of multiplicity \( m \). In other words, the multiplicity is the exponent of the factor.

In the example above, what are the multiplicities of each of the zeros of \( P(x) \)?

Graph \( P(x) = x(x + 5)^2(x - 1)^3 \) and sketch the graph below. What do you notice about the behavior of the graph around the x-intercepts?

In general, for a real zero, the multiplicity of the zero determines the behavior of the graph around that zero.

<table>
<thead>
<tr>
<th>multiplicity of the real zero ( k )</th>
<th>graph around x-intercept of ( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>any even number</td>
</tr>
<tr>
<td>any even number</td>
<td>any odd number &gt; 1</td>
</tr>
</tbody>
</table>

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Definition 21.25. Adding up the multiplicities of the zeros is called **counting the zeros with multiplicity**.

For the same function \( P(x) = x(x + 5)^2(x - 1)^3 \), how many distinct (different) zeros does it have? ______________________ How many zeros does it have, counting with multiplicity? ______________ What is the degree of the polynomial? ______________

**Theorem 21.26.** (Number of Zeros Theorem) Any polynomial with degree \( n \) has exactly __________ complex zeros counting multiplicity. (Note: complex zeros include real zeros.)

**Example 21.27.** Write down any 4th degree polynomial. How many zeros does it have counting multiplicity?

**Example 21.28.** Can you think of a 4th degree polynomial with
- 4 distinct zeros?
- 3 distinct zeros?
- 2 distinct zeros?
- 1 distinct zero?
- 0 distinct zeros?

**Theorem 21.29.** (Number of Distinct Zeros) A polynomial of degree \( n \) has at at most __________ distinct zeros and at least __________ distinct zeros.

*That is, degree (circle one) > ≥ < ≤ number of distinct zeros*

**Theorem 21.30.** (Turning Point Theorem) The graph of a polynomial of degree \( n \) has at most __________ turning points.

*That is, degree (circle one) > ≥ < ≤ number of turning points*

**Example 21.31.** How many non-real complex zeros can a cubic function have? How many real zeros can it have, counting multiplicity?

**Example 21.32.** How many non-real complex zeros can a degree 5 polynomial have? How many real zeros can it have, counting multiplicity?
23 Class 23 and 24 - §4.1, 4.2, and 4.3 - Rational Functions

Definition 23.1. A rational function is a function that can be written as a quotient (ratio) of polynomials:

\[ f(x) = \frac{P(x)}{Q(x)} \]

where \( P(x) \) and \( Q(x) \) are polynomials.

Example 23.2. Some examples of rational functions are:

1. \( f(x) = \frac{4x^3 - 2x + 7}{5x + 1} \)
2. (make one up) \( g(x) = \)

Example 23.3. Two of the simplest and most useful rational functions are \( f(x) = \frac{1}{x} \) and \( g(x) = \frac{1}{x^2} \), graphed below. Identify which graph is which.

You can read more about these functions in §1.2.
Example 23.4. The graph of the function \( h(x) = \frac{4x^2-4}{x^2+3x+10} \) is shown below.

How is the graph of this function \( h(x) \) different from the graph of a polynomial?

Describe the end behavior of this function \( h(x) \).

Definition 23.5. The line \( y = b \) is a horizontal asymptote (HA) for a function \( f(x) \) if the graph of the function approaches the line \( y = b \) as \( x \) gets very large or very negative. In mathematical notation, we write

\[ f(x) \to b \text{ as } x \to \infty \]

or

\[ f(x) \to b \text{ as } x \to -\infty \]

Where does the above function \( h(x) \) have horizontal asymptotes? Draw them as dotted lines on the graph and give their equations.

What is the behavior of the graph of this function \( h(x) \) near \( x = -5 \) and \( x = 2 \)?
Definition 23.6. The line \( x = a \) is a vertical asymptote (VA) for a function \( f(x) \) if the graph of the function soars up towards infinity or plunges down towards negative infinity as \( x \) gets close to \( a \). In mathematical notation, we write

\[
f(x) \to \infty \text{ or } f(x) \to -\infty \text{ as } x \to a^+\]

and

\[
f(x) \to \infty \text{ or } f(x) \to -\infty \text{ as } x \to a^-\]

The notation \( a^+ \) means that \( x \) is approaching \( a \) from the right, and \( a^- \) means \( x \) is approaching \( a \) from the left.

Where does this same function \( h(x) \) have vertical asymptotes? Draw them as dotted lines on the graph and give their equations.

Describe asymptotes in your own words.

What is the domain of this same function \( h(x) \)?

What are the x-intercepts of \( h(x) \)?
Example 23.7. Consider the function

\[ \ell(x) = \frac{x^2 + 8x + 16}{x^2 - x - 30} \]

Find its

- domain
- x-intercepts
- y-intercept
- vertical asymptotes
- horizontal asymptotes

Try to find them algebraically, based on the equation. Then check your work using a graph.
Example 23.8. Find the horizontal asymptotes for the following functions by looking at the ratio of leading terms:

a) \( a(x) = \frac{5x^3 - 3x^2 + 2}{3x^3 + 4x^2 + x} \)

b) \( b(x) = \frac{7x^3 - 6x^2 + 7}{2x^2 + 7x} \)

c) \( c(x) = \frac{8x^3 - 9x + 2}{15x^5 + 7x^3} \)

Note 23.9. Summary: to find the features of a rational function, use the following tricks:

- domain

- \( x \)-intercepts

- \( y \)-intercept

- vertical asymptotes

- horizontal asymptotes
Example 23.10. (optional) Find the domain, vertical asymptotes, horizontal asymptotes, and x-intercepts and graph the function:

\[ q(x) = \frac{x + 1}{2x^2 + 5x - 3} \]

Example 23.11. Match the equations with the graphs on the following page:

A. \[ y = \frac{(x+2)(x-3)}{(x+4)^2} \]
B. \[ y = \frac{(x-3)(x+2)}{2x+4} \]
C. \[ y = \frac{(x+2)^2}{(x+4)^2} \]
D. \[ y = \frac{(x+2)}{(x+4)(x-1)} \]
E. \[ y = \frac{(x+2)^2}{(x+4)(x-1)} \]
F. \[ y = \frac{(x-3)(x+2)}{(x+4)(x-1)} \]
Section 4.3 on rational equations starts here.

**Example 23.12.** What is the least common denominator (LCD) of the fractions \( \frac{7}{15}, \frac{5}{12}, \frac{13}{10} \)?

**Example 23.13.** What is the least common denominator (LCD) of the rational functions \( \frac{1}{x+2}, \frac{x}{x^2+4x+2}, \) and \( \frac{5-x}{x^2+7x+10} \)?

**Note 23.14.** To find the LCD of any expression, factor the denominators, list all the factors that appear, and raise each of them to the highest power that you see the factor raised to.

**Example 23.15.** Solve for \( x \). Hint: “clear the denominators” by multiplying both sides by the LCD.

\[
\frac{x}{x-3} + \frac{1}{x+2} = \frac{18}{x^2 - 9}
\]
Note 23.16. Whenever solving rational equations, it is important to check your answers at the end and eliminate any extraneous solutions. Extraneous solutions are solutions that don’t work when you plug them into the original equation. For rational equations, extraneous solutions are solutions that make a denominator zero.

In the previous example, which answer was an extraneous solution and which was an actual solution?

Example 23.17. Solve for x. Hint: rewrite the negative exponents as fractions.

\[-6x^{-2} + 7x^{-1} = 2\]

Example 23.18. (if time) Solve for x:

\[7x^{-4} - 8x^{-2} + 1 = 0\]