Example 1.1. To rent a U-haul costs $19.95 plus 99 cents per mile. We want to describe cost as a function of the number of miles, in several different ways.

We’ll let the variable $d$ represent distance in miles and $c$ represent cost in dollars.

In function notation, we can write: $c = f(d)$.

$d$ is called the independent variable, input, or “argument”.

$c$ is called the dependent variable, or output.

Fill in the numerical table:

<table>
<thead>
<tr>
<th>miles: $d$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>cost: $c$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Sketch a graph of cost as a function of distance.

Write an equation for cost as a function of distance.

Rewriting the equation using the variables $x$ and $y$ may look more familiar, but it is good to get used to using a variety of letters instead of just $x$ and $y$. In fact, sometimes different letters can help you remember what the variables stand for ($d$ for distance, $c$ for cost).
Definition 1.2. The domain of a function is all input values (x-values) that yield an output value. The range or a function is all output values (y-values).

Question 1.3. What is the domain and range of our function \( f(x) = 19.95 + 0.99x \), as an abstract function?

What is the domain and range of our function \( f(x) = 19.95 + 0.99x \), in the context of the problem?

Example 1.4. Suppose you are moving across town, 5 miles away. Your old house is 3 miles from the U-haul rental place and the new house is 4 miles away, so that your old house, your new house, and the U-haul place form a triangle. If you have to make \( n \) trips with the U-hauls (\( n \) loads of stuff), then how many miles will you travel?

Work out a table of values.

<table>
<thead>
<tr>
<th>number of trips: ( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>distance: ( d )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is the equation for distance as a function of number of trips?
Topic 1.5. Composition of function

**Question 1.6.** We have represented cost as a function of distance and distance as a function of number of loads. How can we represent cost as a function of number of loads?

![Diagram of function composition]

Composition: \( f \circ g(n) = f(g(n)) \) represents cost as a function of number of loads.

**Question 1.7.** How do we compute \( f \circ g(n) \)?

\[
\begin{align*}
f(g(n)) &= f(10n + 2) \\
&= 19.95 + 0.99(10n + 2) \\
&= 19.95 + 9.9n + 1.98 \\
&= 21.93 + 9.9n
\end{align*}
\]

Topic 1.8. Inverses

**Question 1.9.** What if I want to show how far I can go as a function of how much money I have?
We write \( c = f(d) \), and \( d = f^{-1}(c) \).

**Question 1.10.** How do we find a formula for \( f^{-1} \) from the formula for \( f \)?

Method 1: convert to \( x \) and \( y \), reverse the role of \( y \) and \( x \) and solve for \( y \)

\[
\begin{align*}
  y &= 19.95 + 0.99x \\
  x &= 19.95 + 0.99y \\
  x - 19.95 &= 0.99y \\
  \frac{x - 19.95}{0.99} &= y \\
  y &= \frac{x - 19.95}{0.99} \\
  f^{-1}(x) &= \frac{x - 19.95}{0.99}
\end{align*}
\]
Method 2: solve for the “other” variable

\[
c = 19.95 + 0.99d
\]

\[
c - 19.95 = 0.99d
\]

\[
\frac{c - 19.95}{0.99} = d
\]

\[
d = \frac{c - 19.95}{0.99}
\]

\[
f^{-1}(c) = \frac{c - 19.95}{0.99}
\]

Either way, we’ve found \( f^{-1} \) which represents distance as a function of cost.

**Question 1.11.** What is \( f(45) \)? What does it mean?

**Question 1.12.** What is \( f^{-1}(100) \)? What does it mean?
1. Important facts about inverse functions

Example 1.4. \( f(x) = x^3 - 7 \)

a) Find \( f^{-1}(x) \)

b) Find \( f \circ f^{-1}(x) \) and \( f^{-1} \circ f(x) \)

Fact 1.5. \( f \) and \( f^{-1} \) undo each other: \( f \circ f^{-1}(x) = x \) and \( f^{-1} \circ f(x) = x \)

Graph \( f(x) \) and \( f^{-1}(x) \). What do you notice?

Fact 1.6. Graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) are reflections of each other over the line \( y = x \).

Question 1.7. Does every function have an inverse that is a function?

Example 1.8. Consider the function \( f(x) = x^2 \)

\[
\begin{align*}
  f(x) &= x^2 \\
  f(4) &= 16 \\
  f(-4) &= 16
\end{align*}
\]

If \( f \) has an inverse function, what is \( f^{-1}(16) \)? Is it 4 or -4? It is not well-defined, so there is no inverse function for \( f(x) \). You can also see this from the graphs: the flip of the parabola \( y = x^2 \) is not a function because it doesn’t satisfy the vertical line test.
Fact 1.19. A function \( f(x) \) is invertible if and only if the graph of \( y = f(x) \) satisfies the horizontal line test (H.L.T.) A function whose graph satisfies the HLT is called a \textit{one-to-one} function. One-to-one functions have exactly one \( y \)-value corresponding to each \( x \)-value and exactly one \( x \)-value paired with each \( y \)-value.

Example 1.20. \( g(x) = \frac{5 - x}{3x + 2} \)

a) Find \( g^{-1}(x) \)

b) Find the domain and range of \( g \) and \( g^{-1} \).

Fact 1.21. For any function \( g \) with an inverse function \( g^{-1}(x) \), the domain of \( g \) is the range of \( g^{-1} \) and the range of \( g \) is the domain of \( g^{-1} \).

Example 1.22. \( h(x) = 3 \sqrt{7 - 2x + 1} \)

a) Find \( h^{-1}(x) \)

b) Find the domain and range of \( h \) and \( h^{-1} \).
Example 2.1. You are hired for a job and the starting salary is $40,000 with an annual raise of 3% per year. How much will your salary be after 10 years?

First make a guess, without trying to solve the problem exactly. Guess: __________

Note that after one year, you have gotten a 3% raise, so your salary will be $40,000 \times (1.03) = 41,200$. After two years, your salary will be $41,200 \times (1.03) = 42,436$. Continue this reasoning to fill out the table below.

<table>
<thead>
<tr>
<th>years elapsed since starting the job</th>
<th>salary</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>40,000</td>
</tr>
<tr>
<td>1</td>
<td>$40,000 \times (1.03) = 41,200$</td>
</tr>
<tr>
<td>2</td>
<td>42,436</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
Example 2.2. Suppose your starting salary is unknown (just call it $A$), but you know that you'll get the 3% raise every year. How much will your salary be after $1, 2, 3, \ldots, 10$, or $t$ years, in terms of $A$? Try to write a general formula.

<table>
<thead>
<tr>
<th>years elapsed since starting the job</th>
<th>salary</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$A$</td>
</tr>
<tr>
<td>1</td>
<td>$A \cdot (1.03)$</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td></td>
</tr>
</tbody>
</table>

Definition 2.3. A function of the form $f(t) = A \cdot b^t$ for $b > 0$ is called an exponential function.

$A$ represents the initial value.

$b$ is called the growth factor

If we write

$$b = 1 + r$$
$$r = b - 1$$

$r$ is called the growth rate.
Why is it necessary that $b > 0$? What goes wrong, for example, if $b = -2$?

**Example 2.4.** In the first salary example, find:

\[
\begin{align*}
\text{the initial value } A &= \\
\text{the growth factor } b &= \\
\text{the growth rate } r &= 
\end{align*}
\]

**Note 2.5.** The growth rate can be written as a percent or as a decimal. If the growth rate is written as a percent, then the units of growth rate are percent per year or percent per some other time period. If the growth rate is written as a decimal, then the units are just per year or per some other time period.

**Example 2.6.** Fill out the following table

<table>
<thead>
<tr>
<th>growth rate $r$ as a percentage</th>
<th>growth rate $r$ as a decimal</th>
<th>growth factor $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3%</td>
<td>0.03</td>
<td>1.03</td>
</tr>
<tr>
<td>10%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-5%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-30%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>78%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>150%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 2.7.** Suppose your starting salary is $60,000 and you get an annual raise of 5%. Write an equation for your salary after $t$ years.
Example 2.8. The United Nations estimated the world population in 2010 was 6.79 billion, growing at a rate of 1.1% per year. Assume that the growth rate stays the same.

Write an equation for the population at $t$ years after the year 2010.

Use your equation to predict the world population in 2012.

Predict the world population in 2050.

Predict the world population in 2100.

Look online to find the actual population in 2012, and to get other predictions of world population in 2050 and 2100. How do your predictions compare?
Do you think that the growth rate will actually stay at 1.1% or do you think it will go up or down in the future?

Example 2.9. Seroquel is metabolized and eliminated from the body at a rate of 11% per hour. If 400 mg are given, how much remains in the body after 24 hours? (Start out by writing down an equation!)

Example 2.10. After a dose of diazepam is given, the amount left in the body is given by

\[ D(t) = 10 \cdot 0.707^t \]

where \( t \) is time in days and \( D(t) \) is amount in mg. What was the original dose?

What percent leaves the body each day?
Example 2.11. Ron and Hermione invest money in the stock market. The value of Ron’s investment is
\[ R(t) = 200 \cdot (0.87)^t \]
and the value of Hermione’s investment is
\[ H(t) = 100 \cdot (1.01)^t \]
where \( t \) is time in days after investing the money and \( R(t) \) and \( H(t) \) represent money in galleons.

How much did Ron and Hermione each invest?
Are their investments growing or shrinking in value?
By what percent each day?

Insert the symbols \( > \) and \( < \), then numbers 0 and 1, and the words increasing and decreasing into the blanks below.

Fact 2.12. An exponential function \( P(t) = A \cdot b^t \) can also be written as \( P(t) = A \cdot (1+r)^t \), where \( b = 1 + r \). Recall that \( b \) is called the growth factor and \( r \) is called the growth rate. If \( r ____ ____ \), then the quantity \( P(t) \) is growing, that is, \( P(t) \) is an increasing function. If \( r ____ ____ \), then the quantity \( P(t) \) is shrinking (or “decaying”), that is, \( P(t) \) is a decreasing function. In terms of the growth factor \( b \), if \( b ____ ____ \), then the function \( P(t) \) is _________, and if \( b ____ ____ \), the function \( P(t) \) is _________.

§4.2 starts here

Example 2.13. This is a tale of two cities. City A has a population of 50,000 and is increasing by 3000 people each year. City B has a population of 40,000 and is increasing by 5% each year.

Write down a function \( A(t) \) to describe City A’s population over time and a function \( B(t) \) to describe City B’s population over time.

City A is experiencing _________ growth, and City B is experiencing _________ growth.
Fill in the following table.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A(t)$</th>
<th>$B(t)$</th>
<th>differences (for $A(t)$)</th>
<th>differences (for $B(t)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50 000</td>
<td>40 000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>53 000</td>
<td>42 000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>56 000</td>
<td>44 100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What do you notice about the consecutive differences for $A(t)$?

What about the consecutive differences for $B(t)$?

The consecutive differences for $B(t)$ are not constant (that is, they are not the same as each other). Instead of subtracting consecutive numbers for $B(t)$, what else could you do to consecutive numbers, that gives you a constant answer?

**Fact 2.14.** The function $f$ is linear if the ____________ of successive $y$ values is constant.
The function $f$ is exponential if the ____________ of successive $y$ values is constant.

Do you think the population of City B will ever exceed the population of City A? If no, why not? If yes, when? Hint: compare graphs or tables of values.
Example 2.15. Which tables represent linear function and which represent exponential functions?

<table>
<thead>
<tr>
<th>x</th>
<th>3</th>
<th>7</th>
<th>11</th>
<th>15</th>
<th>19</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>1139.6</td>
<td>1051.1</td>
<td>969.53</td>
<td>894.26</td>
<td>824.84</td>
<td>760.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>6</th>
<th>11</th>
<th>16</th>
<th>21</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>g(x)</td>
<td>672.5</td>
<td>637.3</td>
<td>602.1</td>
<td>560.9</td>
<td>531.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>6</th>
<th>10.3</th>
<th>14.6</th>
<th>18.9</th>
<th>23.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>h(x)</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
</tr>
</tbody>
</table>

Our next job is to find equations for these functions.

Consider the equation for a line $y = mx + b$. A parameter is a variable that is fixed for a particular line, such as $m$, the slope. In contrast, a variable like $x$ can take on many different values even for a particular line.

In the equation for a linear function, $y = mx + b$, how many parameters are there? __________ How many points are needed to solve for these parameters? __________

In the general equation for an exponential function, $y = A \cdot b^t$, how many parameters are there? __________ How many points do you think will be needed to solve for these parameters?
a) Find the equation for \( f(x) \). Hint: start by plugging in two points into the equation. Use these points to solve for \( A \) and \( b \).

b) Find the equation for \( g(x) \).

c) Find the equation for \( h(x) \).
Example 2.16. Example 1 from textbook: At time $t = 0$, a species of snake is released onto an island that was snake free. Four years later, there are 300 snakes on the island. Three years after that, there are 450 snakes. Let $S$ represent the snake population and $t$ represent the time in years since the release of the snakes. Find a formula for $P(t)$ assuming:

a) linear population growth

b) exponential population growth

If there are 900 snakes at year 12, which population model is better?
Example 2.17. Example 3 from textbook: The population of a country is initially 2 million people and is increasing at 4% per year. The country’s annual food supply is initially adequate for 4 million people and is increasing at a constant rate adequate for an additional 0.5 million people per year.

a) Based on these assumptions, in approximately what year will this country first experience shortages of food?

b) If the country doubled its initial food supply, would shortages still occur? If so, when? (Assume the other conditions do not change.)

c) If the country doubled the rate at which its food supply increases, in addition to doubling its initial food supply, would shortages still occur? If so, when? (Again, assume the other conditions do not change.)
3 Class 3 - §4.3 and §4.4 - Exponential graphs and compound interest

§4.3 Graphs of exponential functions

Example 3.1. Graph the following functions simultaneously on your calculator and sketch the graphs below. Be sure to label which graph is which.

a) \( y = 2^x \)

b) \( y = 1.5 \cdot 2^x \)

c) \( y = 5 \cdot 2^x \)

Question 3.2. Consider the exponential function \( y = A \cdot b^x \). How does the graph change when you change the initial value \( A \)?

Example 3.3. Without using your calculator, sketch the graph of \( y = 3.5 \cdot 2^x \) on the same axes above. Check your answer with your calculator.
Example 3.4. Graph the following functions simultaneously on your calculator and sketch the graphs below. Be sure to label which graph is which.

a) \( y = 3 \cdot 2.7^x \)
b) \( y = 3 \cdot 1.04^x \)
c) \( y = 3 \cdot 0.65^x \)

Question 3.5. Consider the exponential function \( y = A \cdot b^x \). How does the graph change when you change the growth factor \( b \)?
Example 3.6. Match the equations to the graphs.

Fact 3.7. In the graph of \( y = A \cdot b^x \), the parameter \( A \) gives the y-intercept. The parameter \( b \) tells how the graph is increasing or decreasing. If \( b > 1 \), the graph is \( \text{increasing} \), and if \( b < 1 \), the graph is \( \text{decreasing} \). The closer \( b \) is to the number \( 1 \), the flatter the graph. The farther \( b \) is from the number \( 1 \), the faster (more steeply) the graph is increasing or decreasing. So, for example, the graph of \( y = 0.25^x \) is decreasing (circle one) faster / slower than the graph of \( y = 0.4^x \).

Fact 3.8. All exponential graphs have horizontal asymptotes at the line \( y = 0 \).
Note 3.9. We can use limit notation from calculus to describe horizontal asymptotes. For example, since the graph of \( y = 4 \cdot \left(\frac{1}{3}\right)^x \) has a horizontal asymptote at \( y = 0 \) and approaches it as \( x \) gets very big, we say that \( y \to 0 \) as \( x \to \infty \). Replacing \( y \) with its formula, we can write: \( 4 \cdot \left(\frac{1}{3}\right)^x \to 0 \) as \( x \to \infty \). In limit notation, we write:

\[
\lim_{x \to \infty} 4 \cdot \left(\frac{1}{3}\right)^x = 0.
\]

Similarly, since \( y = 4 \cdot 3^x \to 0 \) as \( x \to -\infty \), we write

\[
\lim_{x \to -\infty} 4 \cdot 3^x = 0.
\]

Note the \( -\infty \) instead of the \( \infty \) sign in the second example.

Note 3.10. We can write statements about end behavior in limit notation even if the end behavior doesn’t involve horizontal asymptotes. For example, the statement

\[
\lim_{x \to \infty} 4 \cdot 3^x = \infty
\]

is a true statement written in limit notation, but doesn’t say anything about the horizontal asymptote, since the function approaches the horizontal asymptote as \( x \to -\infty \), not as \( x \to \infty \).

Example 3.11. Find the horizontal asymptotes of the functions and then express them using limit notation:

<table>
<thead>
<tr>
<th>function</th>
<th>horizontal asymptote</th>
<th>limit notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) ( y = 15 \cdot 5.2^x )</td>
<td>( y = 15 \cdot \frac{1}{5.2} )</td>
<td>( \lim_{x \to \infty} 15 \cdot 5.2^x = 0 )</td>
</tr>
<tr>
<td>b) ( y = -4 \cdot 3^x )</td>
<td>( y = -4 )</td>
<td>( \lim_{x \to -\infty} -4 \cdot 3^x = 0 )</td>
</tr>
<tr>
<td>c) ( y = 2.1 \cdot 0.4^x + 7.2 )</td>
<td>( y = 7.2 )</td>
<td>( \lim_{x \to \infty} 2.1 \cdot 0.4^x + 7.2 = 7.2 )</td>
</tr>
</tbody>
</table>
Example 3.12. Google “population of United States”. You should get a graph that looks something like this:

Would you say that the population growth of the US looks more linear or more exponential? Google the population of other countries until you find a population that looks like it is growing exponentially. Estimate an equation for the population. Hint: when you hold your mouse over the graph, the coordinates of the points appear. Once you have the equation, plot the points and the equation simultaneously on your calculator and see how well the curve fits! You’ll need to use a list and stat plot to plot the points. Ask for help if you need it. Draw a sketch of the graph and write the equation below.
§4.4 Compound Interest

Example 3.13. Read the following dialog and fill in the blanks.

- Senior Manager: You know, all the banks in town are charging 5% interest every year, just like we do. A year is a long time to wait for interest on the money you deposit, and I bet we could attract more customers if we made interest payments more often ... like every month.

- Junior Manager: Let’s do it!

- Senior Manager: All right! We’ll run a big advertising campaign offering 5% interest every month.

- Junior Manager: Great! ..... Uh, wait a sec, I just thought of something. If we offer 5% interest every month, then when someone invests money, their money will grow by a factor of 1.05 every month. So after 2 months, there money will grow by a factor of (1.05)^2. And after a year, it will grow by a factor of (1.05)—. So if someone invests $100, then they’ll have $100 · (1.05)— after a year. That’s $ _____________. In other words about $ ____________ more than they originally invested, which is ____________% interest on the $100. Can we really afford to do that?

- Senior Manager: Oh, no, definitely not. (pause) Too bad. I really wanted to offer interest every month.

- Junior Manager: So how about we chop the 5% interest into 12 pieces, and offer a twelfth of that every month. 5% divided by 12 is 0.41667%. So if we use that interest rate every month, we can offer interest every month and still only pay 5% a year. We’ll call it 5% nominal interest, compounded monthly.

- Janitor: Hang on, I couldn’t help but overhear your conversation. If you offer a twelfth of the 5% every month, you’ll actually end up paying slightly more than 5% a year.

- Managers: Really?

- Janitor: Sure. Since your monthly interest is 0.05/12, your growth factor each month is (1+0.05/12). So each month you multiply whatever is in the account by (1+0.05/12). Since there are 12 months in a year, after a year, you’ve multiplied the original deposit by that factor 12 times, and which amounts to a factor of (1 + 0.05/12)—, or ____________(write down at least four decimal places).

- Senior Manager: Oh ... so if someone deposits $100, how much will they have after a year?
Janitor: Well, it’s $100· ______________, so about $105.12.

Junior Manager: Oh... that sounds like ______________% interest instead of just 5% interest.

Janitor: Exactly.

Senior Manager: I think we can afford that. But we need some catchy phrase to describe how the nominal interest rate of 5%, compounded every month, has the same effect as a 5.12% interest rate.

Janitor: You could say that the effective annual interest rate is 5.12%. Some people also call it APY, for annual percentage yield.

**Definition 3.14.** A *nominal annual interest rate* of $R$ compounded $n$ times a year means that $\frac{R}{n}$ times the current balance is added to the balance $n$ times per year. The nominal annual interest rate is also called the *APR*, for “annual percentage rate”.

**Definition 3.15.** The percent by which an account has actually grown after 1 year is called the *effective annual interest rate*, or the “annual percentage yield” (APY).

**Question 3.16.** In the dialog, the APR is 5%. The money is compounded _____________ times per year, and the APY is _____________.

**Notation 3.17.** I will use a capital $R$ to denote APR, and I will continue to use the lower case $r$ to mean growth rate, as in the salary and population examples previously. Be careful, because the book uses lower case $r$ for both quantities, which are not always the same.

**Question 3.18.** If you have a nominal annual interest rate of 18% compounded monthly, and you invest $100, write a formula for how much money you have at the end of each of the following time periods. You do not need to compute the actual dollar amount, just give the equation.

a) one month?

b) five months?

c) one year?

d) two years?

e) three years?

f) ten years?

g) $t$ years?
Write a formula for how much money you will have if you invest \( A \) dollars at an 18% APR compounded monthly for \( t \) years.

Write a formula for how much money you will have if you invest \( A \) dollars at an APR of \( R \) compounded monthly for \( t \) years.

Write a formula for how much money you will have if you invest \( A \) dollars at an APR of \( R \) compounded weekly for \( t \) years. Hint: there are 52 weeks in a year.

Write a formula for how much money you will have if you invest \( A \) dollars at an APR of \( R \) compounded \( n \) times a year for \( t \) years. (Put a box around this formula – it is the general formula for compound interest.)

**Example 3.19.** You invest $500 at an 8% nominal annual interest rate, compounded daily. How much money will the account earn

a) after 1 year?

b) after 3 years?

What is the daily growth rate of the money in this example? (Write it as a formula rather than computing the actual decimal.)

What is the daily growth factor? (Write it as a formula rather than computing the actual decimal.)
What is the annual growth factor for the money in this example? In other words, what number is the money multiplied by after one year? (Write it as a formula and compute the actual decimal.)

What is the annual growth rate for the money? Hint: the annual growth rate $r = b - 1$.

What percent does the money actually grow by, after a year?

What is the APY?

**Fact 3.20.** The APY is *(circle one)* lower / higher than the APR.

Why is this true?

**Example 3.21.** A Bank account has a 4% APR, compounded weekly. What is its APY?
Example 3.22. Go to a bank website. Find a page that gives rates for vehicle loans (or rates for high yield CD’s or savings accounts) and copy down the APR. Calculate the APY. If the compounding period is not stated, assume that interest is compounded monthly.

Note 3.23. Currently, APR’s are very low on most checking and savings account, often less than 1%. When APR is low, then APY and the APR are almost the same. To see a difference between APR and APY, you’ll have to look for accounts with a high interest rate, like long term CD’s or look at loans, such as vehicle loans. Mortgages are more complicated, so don’t use those.

Note 3.24. At the Redwood Credit Union website, the page on auto loans lists APR’s (but not APY’s), while the page on savings accounts lists APY’s (but not APR’s). Why do you think the credit union might present the information this way?
4 Class 4 - §4.5 and §5.1 - Continuous compounding and log review

In the previous section, we found that an investment of $A$ dollars at an APR of $R$, compounded $n$ times per year, yields

$$P(t) = A \cdot \left(1 + \frac{R}{n}\right)^{nt}$$
dollars after $t$ years.

**Question 4.1.** If you invest $1000, which gives you more money after a year, 5% APR compounding monthly or 5% APR compounded daily? __________

Which gives you more money after a year: 5% APR compounded daily or 5% compounded every hour? __________

**Question 4.2.** As $n$, the number of compounding periods per year, gets bigger and bigger, what do you think happens to the amount of money earned after a year? Could we make $1000 grow to $1 million, just by compounding often enough, say, a billion times a second?

**Question 4.3.** To phrase this question more mathematically, what happens to $P(t) = A \cdot (1 + \frac{R}{n})^{nt}$ as $n \to \infty$?

For simplicity, let’s start out by assuming that $A = 1$, $r = 100\%$, and $t = 1$. So $P(t) = (1 + \frac{1}{n})^n$

**Example 4.4.** Fill in the table, writing down at least 6 decimal places:

<table>
<thead>
<tr>
<th>$n$</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10,000</th>
<th>100,000</th>
<th>1,000,000</th>
<th>10,000,000</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(t) = (1 + \frac{1}{n})^n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Fact 4.5.** As $n \to \infty$, $P(t) = (1 + \frac{1}{n})^n \to __________$. In other words, if you invest $1$ at 100% APR, compounded every instant (i.e. “continuously”), then at the end of a year, you have __________ dollars. (Hint: you may recognize the digits in the table as digits of a very famous number.)

**Definition 4.6.** Continuous compounding is the limit of compounding more and more frequently, at shorter and shorter intervals; that is, the limit as the number of compounding periods per year goes to infinity.

**Example 4.7.** Fill in a similar table for an APR of 5% instead of 1%, to find out how much money you have if you invest $1 for 1 year at 5% APR compounded continuously.

<table>
<thead>
<tr>
<th>$n$</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10,000</th>
<th>100,000</th>
<th>1,000,000</th>
<th>10,000,000</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(t) = (1 + \frac{0.05}{n})^n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fact 4.8. As \( n \to \infty \), \( P(t) = (1 + \frac{0.05}{n})^n \to \) ____________ . In other words, if you invest $1 at 5% APR, compounded continuously, then at the end of a year, you have ____________ dollars. (Hint: The answer is related to the previous answer about 100% interest, but you have to work the 0.05 in there somehow, in a way that reproduces the numbers you see in the table.)

Question 4.9. If, instead of one dollar, you invest \( A \) dollars at 5% APR, compounded continuously, then at the end of a year, how much money do you have? ____________

Fill in the following table with formulas, to show what happens to \( A \) dollars invested at 5% APR compounded continuously.

<table>
<thead>
<tr>
<th>years</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>money</td>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fact 4.10. If you invest \( A \) dollars at an APR of \( R \), compounded continuously, after \( t \) years, you have ____________ dollars. (give the formula).

Example 4.11. If you deposit $400 in a bank account that earns 8% nominal annual interest, compounded continuously,

a) how much money will you have after 5 years?

b) what is the effective annual interest rate?

Example 4.12. Give an equation for the money you have after \( t \) years, if you deposit $200 in an account with a 12% APR that is

a) compounded annually

b) compounded monthly

c) compounded continuously
Example 4.13. Graph the three equations in the previous example.

a) which of the three graphs look like the graphs of exponential functions?

b) which graph is growing the fastest?

c) what are the initial values?

d) what are the growth factors? (Hint: rewrite each equation in the form \( y = A \cdot b^t \) using your exponent rules.)

e) what are the growth rates?

Note 4.14. In the previous example, you may have noticed that the function \( y = 200 \cdot e^{0.12t} \) is an exponential function. Even though it is not given in the form \( y = A \cdot b^t \), it can be rewritten in that form. In fact, any function of the form \( y = A \cdot e^{kt} \) is an exponential function and can be rewritten in the form \( y = A \cdot b^t \) for some number \( b \).

Definition 4.15. A function of the form \( f(t) = A \cdot e^{kt} \) is called an exponential function in continuous form. The number \( k \) is called the continuous growth rate.

Definition 4.16. A function of the form \( f(t) = A \cdot b^t \) is called an exponential function in general form.

Fact 4.17. A function of the form \( f(t) = A \cdot e^{kt} \) can be rewritten in the form \( f(t) = A \cdot b^t \) by rewriting it and letting \( b = \text{________} \).
Example 4.18. Write equations to describe each of the following populations:

a) A population grows at a **continuous rate** of 6% per year.

b) A population shrinks at a **continuous rate** of 10.5% per year.

Example 4.19. Find the growth factors for each of these two populations. Hint: use exponent rules to rewrite the equations to look more like $y = A \cdot b^t$.

a) .

b) .

Example 4.20. Find the growth rates for each of these two populations and compare them to the continuous growth rates that you were given originally.

a) .

b) .
Example 4.21. Which grows faster?

1. a population that grows at a continuous rate of 7% per year
2. a population that grows at a rate of 7% per year

Example 4.22. Which decreases faster?

1. a medication that decays at a continuous rate of 9% per hour
2. a population that decays at a rate of 9% per hour

Fact 4.23. In practice, for reasonable growth and decay rates, continuous growth rates and regular growth rates are very close to each other. However, a \( k\)\% continuous growth rate is slightly (circle one) slower / faster growth than a \( k\)\% growth rate. A \( k\)\% continuous decay rate is slightly (circle one) slower / faster decay than a \( k\)\% decay rate.

Question 4.24. Can you explain the previous fact using the language of compound interest?
Fact 4.25. Previously, we saw the following relationships for increasing vs. decreasing functions. Add a line for continuous growth rate $k$.

<table>
<thead>
<tr>
<th>increasing function</th>
<th>decreasing function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b &gt; 1$</td>
<td>$b &lt; 1$</td>
</tr>
<tr>
<td>$r &gt; 0$</td>
<td>$r &lt; 0$</td>
</tr>
<tr>
<td>$k$</td>
<td>$k$</td>
</tr>
</tbody>
</table>

Example 4.26. Convert this equation from continuous form to general form.

$$y = 212.6 \cdot e^{-0.13t}$$

Example 4.27. Convert this equation from general to continuous form. (Hint: you will need to use logarithms.)

$$y = 351 \cdot 1.825^t$$
**Note 4.28.** (Optional) People who work with finances and people who study population growth often use different terminology to describe the same mathematical ideas and formulas. Draw lines to match the concept on the left to the corresponding concept on the right. Some correspondences will be approximate only, and some terms on the left may match to the same term on the right.

<table>
<thead>
<tr>
<th>Finances</th>
<th>Population growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>APR, compounded continuously</td>
<td>annual growth rate ( r )</td>
</tr>
<tr>
<td>APY</td>
<td>continuous growth rate ( k )</td>
</tr>
<tr>
<td>effective annual interest</td>
<td>general form of the exponential equation ( y = A \cdot b^t )</td>
</tr>
<tr>
<td>equation for continuous compounding: ( y = A \cdot e^{kt} )</td>
<td>continuous form of the exponential equation ( y = A \cdot e^{kt} )</td>
</tr>
<tr>
<td>( (1 + \frac{R}{n})^n )</td>
<td>formula for compound interest</td>
</tr>
<tr>
<td>( y = A \cdot (1 + \frac{R}{n})^{nt} )</td>
<td></td>
</tr>
</tbody>
</table>
Topic 4.29. Review of Logarithms

**Definition 4.30.** $\log_a b = c$ means ____________.

**Note 4.31.** You can think of logarithms as exponents: $\log_a b$ is the exponent (or “power”) that you have to raise $a$ to, in order to get $b$. The number $a$ is called the base of the logarithm.

**Notation 4.32.** The natural log, written $\ln x$, means $\log_e x$. The common log, written $\log x$, with no base, means $\log_{10} x$.

**Example 4.33.** Rewrite the following statements using exponents instead of logs:

a) $\log_3 \frac{1}{9} = -2$

b) $\log(5 + x) = v$

**Example 4.34.** Rewrite the following statements using logs:

a) $10^{-4} = 0.0001$

b) $5^{3x + 2} = 6z$

**Example 4.35.** Let $f(x) = \log x$. Find $f^{-1}(x)$.
Fact 4.36. $\log_{10} 10^{x} =$ ___________. This is true because:

Fact 4.37. $10^{\log_{10} x} =$ ___________. This is true because:

Fact 4.38. For any base $b$, $\log_{b} b^{x} =$ ___________ and $b^{\log_{b} x} =$ ___________.

Example 4.39. Simplify the following expressions:
   a) $\ln e^{-3}$
   b) $e^{\ln(5+w)}$
   c) $\log \sqrt{10}$
   d) $10^{\log 3z}$

Fact 4.40. Recall some of the exponent laws.
   1. $2^0 = 1$
   2. Product rule: $2^n \cdot 2^m = 2$__________
   3. Quotient rule: $\frac{2^n}{2^m} = 2$__________
   4. Power rule: $(2^n)^m = 2$__________

Note 4.41. The exponent rules hold for any base, not just base 2.

Fact 4.42. The corresponding logarithm rules are:
   1. $\log_{2}$__________ = ____________
   2. Product rule: ________________
   3. Quotient rule: ________________
   4. Power rule: ________________

Note 4.43. The logarithm rules hold for any base, not just base 2.
Proof. of the product rule. For each line in the proof, provide a reason to justify that line.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( r = \log_2 x ) and ( s = \log_2 y ), then ( x = 2^r ) and ( y = 2^s ).</td>
<td></td>
</tr>
<tr>
<td>So ( x \cdot y = 2^r \cdot 2^s ).</td>
<td></td>
</tr>
<tr>
<td>So ( x \cdot y = 2^{r+s} ).</td>
<td></td>
</tr>
<tr>
<td>Therefore, ( \log_2(x \cdot y) = r + s ).</td>
<td></td>
</tr>
<tr>
<td>So ( \log_2(x \cdot y) = \log_2 x + \log_2 y ).</td>
<td></td>
</tr>
</tbody>
</table>

Proof. of the quotient rule. Write down a proof of the quotient rule along the lines of the previous proof of the product rule.

Proof. of the power rule. For each line in the proof, provide a reason to justify that line.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( r = \log_2 x ), then ( x = 2^r ).</td>
<td></td>
</tr>
<tr>
<td>So ( x^m = (2^r)^m ).</td>
<td></td>
</tr>
<tr>
<td>So ( x^m = 2^{rm} ).</td>
<td></td>
</tr>
<tr>
<td>So ( \log_2 x^m = r \cdot m ).</td>
<td></td>
</tr>
<tr>
<td>Therefore, ( \log_2 x^m = m \log_2 x ).</td>
<td></td>
</tr>
</tbody>
</table>
Example 4.44. Use properties of logarithms to get rid of products, quotients, and exponents in the following expressions.

a) \( \log(10x) \)

b) \( \ln \left( \frac{e^3}{\sqrt{x}} \right) \)

Example 4.45. Write the expression as a single log.

a) \( 2 \log 5 + \log 4 \)

b) \( 3(\log x + \frac{4}{3} \log y) \)

Example 4.46. Simplify.

a) \( \log(10^{2x}) \)

b) \( e^{3 \ln x} \)

c) \( \log(3 \cdot 10^{2x}) \)

Example 4.47. Solve: \( 5 \cdot 2^{x+1} = 17 \)
Example 4.48. Solve: $5 \cdot e^{-0.05t} = 3 \cdot e^{0.2t}$

Example 4.49. Solve: $6 \cdot 3^t = 7 \cdot 2^{5t}$

Example 4.50. Solve: $4 \cdot x^6 - 1 = 18$
Example 4.51. Solve: $2 \ln(2x + 5) - 3 = 1$

Example 4.52. Solve: $\log(x + 3) + \log(x) = 1$

Example 4.53. True or False:

1. $\ln(9x + 17) = \ln(9x) + \ln(17)$
2. $\frac{\ln(x^2)}{\ln(x)} = 2$
3. $\ln(x + 1) \cdot \ln(x - 1) = \ln(x^2 - 1)$
4. $\ln(5x^3) = 5 \ln(x^3)$
5. $\ln(w - v) = \frac{\ln(w)}{\ln(v)}$
6. $\ln\left(\frac{1}{a}\right) = -\ln(a)$
5 Class 5 - §5.2 - Half life and doubling time

Example 5.1. Nigeria’s population in 2010 was 158.4 million, and was growing at a rate of 2.5% per year. If this growth rate continues,

a) Predict when the population of Nigeria will reach 200 million.

b) How long will it be until the population of Nigeria doubles?
Example 5.2. Suppose a bacteria culture grows at a rate of 3% per hour. When will the population

a) double?

b) quadruple?

c) reach 8 times its original size?

d) without doing any calculations beyond simple arithmetic, predict how long it would take for the population to reach 16 times its original size.
Definition 5.3. The time it takes for a population to double in size is called its doubling time.

Note 5.4. For exponential growth, the doubling time does not depend on the original population size, only on the growth rate. Why?

Example 5.5. Suppose a bacteria population doubles every 15 minutes.

a) Write an equation for its growth using the general form of the exponential equation \( y = a \cdot b^t \).

b) What is its growth rate?

c) Find an equation for its growth using the continuous form of the exponential equation \( y = a \cdot e^{kt} \).
d) What is its continuous growth rate?

**Note 5.6.** When you are given the doubling time and need to find the equation for growth, you can choose whether to use the continuous form of the exponential equation \( y = A \cdot e^{kt} \) or the general form of the equation \( y = A \cdot b^t \). What different algebra techniques are used to find the parameter \( k \) vs. the parameter \( b \)?

**Definition 5.7.** The **half life** is the time it takes for a substance to decrease to half its original amount.

**Note 5.8.** Like doubling time, half life does not depend on the original amount of the substance, only on its decay rate (for exponential decay). Why?

**Example 5.9.** Tylenol is metabolized and eliminated at a continuous rate of 27.7% per hour. What is its half life?

Compare its half life to the recommended dosage frequency of 4 to 6 hours.

**Example 5.10.** The half life of radioactive Carbon-14 is 5750 years.

a) Write an equation to describe its decay.
b) If you have 200 grams originally, how much will be left after 50 years?

c) A fossil has 20% of its Carbon-14 left. How old is it?

Example 5.11. It is handy to be able to go back and forth between the continuous and the general form of the exponential equation. Fill in the following table.

<table>
<thead>
<tr>
<th>$y = ab^t$</th>
<th>$y = ae^{kt}$</th>
<th>annual growth / decay rate</th>
<th>continuous annual growth / decay rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 5(1.2)^t$</td>
<td></td>
<td>-9%</td>
<td></td>
</tr>
<tr>
<td>$y = 6e^{-0.04t}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Class 6 - §5.3 and 5.4 - Log functions and log scales

§5.3 starts here

Example 6.1. Draw the graph of $y = 10^x$ and the graph of $y = \log x$ on the same axes. Be careful! Your calculator graph of $y = \log x$ may be misleading!

Example 6.2. List the key features of the graph of $y = 10^x$ and $y = \log x$ on the following table. Sketch the asymptotes as dashed lines on the plot you made above.

<table>
<thead>
<tr>
<th>Function</th>
<th>$y = 10^x$</th>
<th>$y = \log x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Range:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Asymptotes:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Describe asymptotes using limit notation:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intercepts:</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Note 6.3. Near vertical asymptotes, sometimes a function approaches infinity on one side of the asymptote and approaches negative infinity on the other side. Also, sometimes the function is just defined on one side of the asymptote, like \( y = \log x \) above. It is possible to specify the behavior of a function on just one side of a vertical asymptote using limit notation by attaching a plus sign as a superscript to mean “on the right side” and attaching a minus sign to mean “on the left side”, as in the example below.

The vertical asymptote at \( x = 2 \) can be described in limit notation by writing:

\[
\lim_{x \to 2^-} = \infty, \quad \lim_{x \to 2^+} = -\infty
\]

Describe the vertical asymptote at \( x = -3 \) using limit notation in a similar way.
Example 6.4. Match the graphs to the equations without using your calculator.

Example 6.5. Let $g(x) = 4e^{x-2} - 7$

a) Find $g^{-1}(x)$

b) Find the domain and range of $g(x)$. 

a) $y = 2^x$

b) $y = e^x$

c) $y = \log_2(x)$

d) $y = \ln(x)$
c) Find the domain and range of \( g^{-1}(x) \).

**Example 6.6.** Let \( h(x) = 5 \ln(1 - 4x) + 9 \)

a) Find \( h^{-1}(x) \)

b) Find the domain and range of \( h(x) \).

c) Find the domain and range of \( h^{-1}(x) \).
Topic 6.7. Decibels

Note 6.8. Sound intensity is measured in watts/cm$^2$. The intensity of audible sound varies over a huge range of values, from a whisper, which is about $10^{-13}$ watts/cm$^2$, to a power saw, which is about $10^{-5}$ watts/cm$^2$ at three feet away.

Example 6.9. How many times louder (more intense) is a power saw than a whisper?

Note 6.10. Because sound intensities vary so much, logarithms are used to convert sound intensities to a unit of measurement called decibels, using the formula:

$$dB = 10 \cdot \log\left(\frac{I}{I_0}\right)$$

where $dB$ means decibels, $I$ represents the intensity of the sound you are interested in, and $I_0$ represents the intensity of the lowest audible sound: $I_0 = 10^{-16}$.

Example 6.11. Fill in the following table for intensity and decibels.

<table>
<thead>
<tr>
<th>sound</th>
<th>whisper</th>
<th>conversation</th>
<th>subway train at 200'</th>
<th>loud rock concert</th>
<th>pain begins</th>
<th>jet engine at 100'</th>
</tr>
</thead>
<tbody>
<tr>
<td>intensity</td>
<td>$10^{-13}$</td>
<td>3.1623 $\times 10^{-5}$</td>
<td></td>
<td></td>
<td></td>
<td>0.01</td>
</tr>
<tr>
<td>dB</td>
<td>60</td>
<td>95</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 6.12. If your sister yells at you at 80 dB and you yell back at 90 dB, how many times louder (more intense) is your yell than your sister’s? Note: one way to solve this is by computing intensities and taking the ratio, but you don’t actually have to compute the intensities if you use log properties to find the ratios in terms of the decibels.
Example 6.13. If a sound doubles in intensity, how many units does its decibel rating increase? Hint: call the original sound $I$. Then a sound with twice the intensity is $2I$. Use the formula for decibels to continue.

§5.4 Log Scales (if time permits)

Note 6.14. Logarithmic scales are used to represent quantities of vastly differing sizes.

Example 6.15. Consider the distances of the planets from the sun:

<table>
<thead>
<tr>
<th>planet</th>
<th>distance in millions of miles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>58</td>
</tr>
<tr>
<td>Venus</td>
<td>108</td>
</tr>
<tr>
<td>Earth</td>
<td>149</td>
</tr>
<tr>
<td>Mars</td>
<td>228</td>
</tr>
<tr>
<td>Jupiter</td>
<td>778</td>
</tr>
<tr>
<td>Saturn</td>
<td>1426</td>
</tr>
<tr>
<td>Uranus</td>
<td>2869</td>
</tr>
<tr>
<td>Neptune</td>
<td>4495</td>
</tr>
</tbody>
</table>

Draw these distances on a number line. Tip: use a ruler or graph paper for better accuracy.
Now, try converting all the distances to their logs. Tip: to save time on your calculator, use STAT → EDIT to enter the list of numbers in list L1 and then set L2 = \log(L1) to compute the logs of all values at once.

<table>
<thead>
<tr>
<th>planet</th>
<th>distance in millions of miles</th>
<th>log of distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>58</td>
<td></td>
</tr>
<tr>
<td>Venus</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>Earth</td>
<td>149</td>
<td></td>
</tr>
<tr>
<td>Mars</td>
<td>228</td>
<td></td>
</tr>
<tr>
<td>Jupiter</td>
<td>778</td>
<td></td>
</tr>
<tr>
<td>Saturn</td>
<td>1426</td>
<td></td>
</tr>
<tr>
<td>Uranus</td>
<td>2869</td>
<td></td>
</tr>
<tr>
<td>Neptune</td>
<td>4495</td>
<td></td>
</tr>
</tbody>
</table>

Plot the logs of the distances on a number line. Notice how the planets are now more evenly spaced. Tip: use a ruler or graph paper for better accuracy.

---

**Note 6.16.** Since the actual distance is more meaningful than the log of the distance, people often label the number line with the actual values, rather than the logs of the values, as shown below. Plot the planets in the same places as right above, but now label the tick marks with actual distances instead of logs of distances.

---

Try using log graph paper to plot the actual distances of the planets (not the logs of the distances). On log graph paper, the evenly spaced lines represent powers of 10 (1, 10, 100, 1000, 10000) and the lines in between that bunch up towards the right side represent intermediate values: for example, the bunching up lines in between 10 and 100 represent the values of 20, 30, 40, 50, 60, 70, 80, and 90.

Compare your plot of actual distances on log graph paper to a plot of logs of distances on regular graph paper.
Example 6.17. Sometimes it is useful to plot two variables both on log scales. Use the STAT and STAT PLOT buttons on your calculator to plot the log of the distance on the $x$ axis and the log of the orbital period on the $y$ axis. Compare this to the plot you get when you just plot distance on the $x$ axis and orbital period on the $y$ axis. What do you notice?

<table>
<thead>
<tr>
<th>planet</th>
<th>distance in millions of miles</th>
<th>log of distance</th>
<th>orbital period in Earth years</th>
<th>log of orbital period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>58</td>
<td></td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>Venus</td>
<td>108</td>
<td></td>
<td>0.62</td>
<td></td>
</tr>
<tr>
<td>Earth</td>
<td>149</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Mars</td>
<td>228</td>
<td></td>
<td>1.88</td>
<td></td>
</tr>
<tr>
<td>Jupiter</td>
<td>778</td>
<td></td>
<td>11.86</td>
<td></td>
</tr>
<tr>
<td>Saturn</td>
<td>1426</td>
<td></td>
<td>29.46</td>
<td></td>
</tr>
<tr>
<td>Uranus</td>
<td>2869</td>
<td></td>
<td>84.01</td>
<td></td>
</tr>
<tr>
<td>Neptune</td>
<td>4495</td>
<td></td>
<td>164.8</td>
<td></td>
</tr>
</tbody>
</table>

You can get the same sort of effect by plotting the $x$ and $y$ values using log-log graph paper.

Example 6.18. Sometimes it is useful to plot just the $y$-variable on a log scale. Try plotting time on the $x$-axis and LP sales on the $y$-axis. Then try plotting time on the $x$-axis and ln(LP sales) on the $y$-axis. What types of functions do the two graphs look like?

<table>
<thead>
<tr>
<th>years since 1982</th>
<th>LP sales</th>
<th>ln(LP sales)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>244</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>210</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>205</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>167</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>107</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>72</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4.8</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2.3</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1.2</td>
<td></td>
</tr>
</tbody>
</table>

Fit a curve to each of the two plots: 1) time vs. LP sales, and 2) time vs. ln(LP sales). Compare the two equations and explain their relationship.
Chapter 7 Skills Refresher starts here

Topic 7.1. Trigonometry is the study of triangles and circles. It was developed by Greek, Indian, Islamic, and Chinese mathematicians from about 300 B.C. on, and was used in navigation, astronomy, engineering, and surveying. It continues to be used today in these fields and more.

Example 7.2. Find the side lengths of a right triangle with angles 45°, 45°, 90° and hypotenuse 1. Hint: use the Pythagorean Theorm: $a^2 + b^2 = c^2$

Example 7.3. Find the side lengths of a right triangle with angles 45°, 45°, 90° and hypotenuse 5.

Example 7.4. Find the side lengths of a right triangle with angles 30°, 60°, 90° and hypotenuse 1. Hint: double the triangle by reflecting it over its longest side.
Example 7.5. Find the side lengths of a right triangle with angles $30^\circ$, $60^\circ$, $90^\circ$ and hypotenuse 5.
Definition 7.6. Two triangles are similar if they have the same shape but not necessarily the same size.

Fact 7.7. In similar triangles, corresponding angles are congruent (have the same measure).

Fact 7.8. In similar triangles, corresponding sides are all in the same proportion.

Example 7.9. Find the missing sides in these similar triangles.
**Definition 7.10.** For an angle $\theta$ in a right triangle that is not the right angle, the functions sine, cosine, and tangent of $\theta$ are defined as follows:

\[
\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \\
\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \\
\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}
\]

**Note 7.11.** You can remember these definitions with the mnemonic “SOHCAHTOA”, which stands for “sine-opposite-hypotenuse-cosine-adjacent-hypotenuse-tangent-opposite-adjacent”.

**Example 7.12.** Find the $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for the triangle below.

**Question 7.13.** Suppose you have two different right triangles with the same angle $\theta$. Suppose Liliana computes $\sin(\theta)$ using the triangle on the left and Rachel computes it using the triangle on the right. Will they get the same answer? Explain.
Example 7.14. Without using a calculator, figure out \( \sin(45^\circ) \), \( \cos(45^\circ) \), and \( \tan(45^\circ) \) using this right triangle with hypotenuse 1.

![Right Triangle with Hypotenuse 1](image)

Example 7.15. Without using a calculator, find \( \sin(45^\circ) \), \( \cos(45^\circ) \), and \( \tan(45^\circ) \) using a right triangle with hypotenuse 5.

![Right Triangle with Hypotenuse 5](image)

Example 7.16. Without using a calculator, find \( \sin(30^\circ) \), \( \cos(30^\circ) \), and \( \tan(30^\circ) \) using this right triangle with hypotenuse 1.

![Right Triangle with Hypotenuse 1](image)

Example 7.17. Without using a calculator, find \( \sin(60^\circ) \), \( \cos(60^\circ) \), and \( \tan(60^\circ) \).
Example 7.18. Summarize your results in this chart:

<table>
<thead>
<tr>
<th>angle $\theta$</th>
<th>$\cos(\theta)$</th>
<th>$\sin(\theta)$</th>
<th>$\tan(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$30^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$45^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$60^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note 7.19. For other angles, it is more difficult to compute an exact answer for $\sin(\theta)$ and $\cos(\theta)$, and we will usually just use a calculator to get a decimal approximation.

Example 7.20. Use your calculator to compute $\sin(40^\circ)$, $\cos(75^\circ)$, and $\tan(14^\circ)$. Make sure that your calculator is in DEGREE mode, and not radian mode, before you do this!

Example 7.21. Find $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for the triangle drawn below.

Example 7.22. Use sine and cosine to find $x$ and $y$ below.
Note 7.23. Up to now we have defined sine, cosine, and tangent in terms of right triangles. For example, to find $\sin(14^\circ)$ we need to take the ratio of $\frac{\text{opposite}}{\text{hypotenuse}}$ for a right triangle that has one angle of measure $14^\circ$.

Question 7.24. Is it possible to find a right triangle with one angle of $14^\circ$? ________________
What angle measures can you not find as the angle in a right triangle? ________________

Note 7.25. If we try to create a right triangle with one angle of $120^\circ$, we run into trouble.

Instead of using a right triangle to define $\sin(120^\circ)$ and $\cos(120^\circ)$, we need to use a unit circle.

Definition 7.26. The unit circle is a circle of radius 1, centered at the origin.

Example 7.27. Write a sentence to explain the relationship between right triangles and the unit circle, based on this picture.
Note 7.28. Embed a right triangle in the unit circle as drawn, and label its top vertex with the coordinates \((a, b)\). In terms of \(a\) and \(b\), how long is the base of the triangle? How long is its height?

\[
\text{base} = \____
\text{height} = \____
\]

Using the triangle definition of sine and cosine, what are \(\sin(\theta)\) and \(\cos(\theta)\) in terms of \(a\) and \(b\)?

\[
\sin(\theta) = \____
\cos(\theta) = \____
\]

Fact 7.29. For right triangles drawn on the unit circle, the triangle definition of sine and cosine show that \(\cos(\theta)\) is the \____-coordinate of the point on the unit circle at angle \(\theta\) and \(\sin(\theta)\) is the \____-coordinate.

Definition 7.30. For angles that can’t be part of a right triangle (because they are not between 0° and 90°), find the point on the unit circle at angle \(\theta\) and define \(\cos(\theta) = x\)-coordinate and \(\sin(\theta) = y\)-coordinate of this point.

Note 7.31. Recall that for right triangles, tangent is defined by \(\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}\). For a right triangle on a unit circle (as drawn above), we can also write tangent in terms of the \(x\)- and \(y\)-coordinates of the point at angle \(\theta\):

\[
\tan(\theta) = \____
\]

Since \(\cos(\theta)\) is the \(x\)-coordinate and \(\sin(\theta)\) is the \(y\)-coordinate, we can also write tangent in terms of cosine and sine as:

\[
\tan(\theta) = \____
\]
Note that the above definitions of tangent work for any angle on the unit circle, not just angles between $0^\circ$ and $90^\circ$ or angles in a right triangle.

**Example 7.32.** For the angle $\phi$ drawn, $\sin(\theta) =$_________ and $\cos(\theta) =$_________ and $\tan(\theta) =$_________.

**Example 7.33.** Without a calculator, find $\sin(120^\circ)$ and $\cos(120^\circ)$. Hint: drop a vertical line segment from the point $P$.

**Notation 7.34.** When using the unit circle, there are some conventions for drawing angles.

1. Measure angles in the counterclockwise direction, starting from the positive x-axis.
2. Negative angle go in the clockwise direction.
3. Angles greater than $360^\circ$ wrap around the circle more than once.
Example 7.35. Use the unit circle to find sine, cosine, and tangent of the following angles, without using a calculator.

<table>
<thead>
<tr>
<th>angle $\theta$</th>
<th>$\cos(\theta)$</th>
<th>$\sin(\theta)$</th>
<th>$\tan(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$180^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-90^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$810^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$225^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-30^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

400° = 360° + 40°
Example 7.36. Without a calculator, fill out the angles and the exact x- and y-coordinates of the points on the blank unit circle. Hint: fill out the first quadrant, and then use symmetry to fill out the other quadrants. The angles in the first quadrant are 30°, 45°, and 60°.
Note 7.37. You will need to memorize the sine and cosine of the “special angles” on this circle, but most of the time (and for the rest of this class period) we will use calculators to find values for other angles.

Example 7.38. Find the coordinates of the point on the unit circle at an angle of 103°.

Example 7.39. Find the coordinates of a point on a circle of radius 7 at an angle of 103°.

Note 7.40. The coordinates of a point on a circle of radius \( r \) at an angle of \( \theta \) are

\[
(\quad , \quad )
\]

Example 7.41. Find an angle on a unit circle with the same cosine as 22°.

Find an angle on a unit circle with the same sine as 22°.
Example 8.1. Use your calculator, or your unit circle from last time, to fill in the following table with decimal values for \( \cos(t) \) and \( \sin(t) \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos(t) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sin(t) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Use the above table to plot points and sketch a graph of \( y = \cos(t) \) and \( y = \sin(t) \). Note that the angle \( t \) goes on the x-axis and the values of \( \cos(t) \) and \( \sin(t) \) go on the y-axis.
Example 8.2. Here are the computer’s versions. Without using your calculator, correctly label one function as \( y = \sin(t) \) and one as \( y = \cos(t) \).

![Graph of trigonometric functions]

Question 8.3. What are the domain and range of \( f(t) = \cos(t) \) and \( g(t) = \sin(t) \)? Explain why range makes sense in terms of the unit circle.

Note 8.4. Notice that \( f(t) = \cos(t) \) and \( g(t) = \sin(t) \) repeat in the horizontal direction. If you shift \( f(t) = \cos(t) \) horizontally left or right by 360, it will line up perfectly with itself. Using transformation notation, \( \cos(t + 360^\circ) = \cos(t) \) and \( \cos(t - 360^\circ) = \cos(t) \).

Definition 8.5. For a function \( f(x) \), if \( f(x + c) = f(x) \) for some number \( c \) (and if \( c \) is the smallest positive number for which this is true), then \( f(x) \) is called **periodic** with period \( c \).

Question 8.6. What is the period of \( \cos(t) \)? ________

What is the period of \( \sin(t) \)? ________.

Use the unit circle to explain why \( \cos(t) \) and \( \sin(t) \) have this period.
§7.4 starts here

**Note 8.7.** Recall that \( \tan(\theta) = \frac{y}{x} \) where \( x \) and \( y \) are the \( x \) and \( y \)-coordinates of a point at angle \( \theta \) on the unit circle.

**Example 8.8.** What is the equation of the line at angle \( \theta \) drawn below, and what does its equation have to do with \( \tan(\theta) \)?

![Diagram of a line on the unit circle with angle \( \theta \).]

**Note 8.9.** In general, the slope of the line at angle \( \theta \) through the origin is ____________.

When the angle is zero, the slope is ______________. As the angle increases toward 90°, the slope goes to _____________. As the angle goes from zero towards −90°, the slope goes to _____________. At exactly 90° and −90°, the slope is ______________.

Use these ideas about slope of the line at angle \( \theta \) to sketch a rough graph of \( y = \tan(t) \) for \( t \) between −90° and 90° below.
Compare your sketch to the graph drawn below.

In the graph above, find the:

- x-intercepts:
- vertical asymptotes:
- domain:
- range:

**Example 8.10.** You need to measure the width of a river that is too dangerous to cross. You set up a survey post directly across the river from some landmark, like a tree. Then you head downstream 400 meters. Now you measure the angle of sight to the tree as 31°. Calculate the width of the river.
Example 8.11. We will hold a rocket building contest to see whose homemade paper rocket goes the highest, using air to blast them off. Design a method to determine the height of a launched rocket, using only a clinometer (to measure angles of sight from the horizontal), measuring tape, and a calculator. Describe your method below, and include figures.

Attached are instructions from the Exploratorium for building a paper rocket. You are free to deviate from the instructions however you like. We will use the stomp rocket launcher to launch rockets; you can use the black tube from the launcher (instead of a PVC pipe) to wrap paper and build your rocket body the right size.
Example 9.1. A 12% grade means that the vertical rise is 12 feet per 100 feet of horizontal distance. Suppose that a road climbs at an angle of 15°. What is its grade?

Example 9.2. What is the angle of a road that has an 8% grade?
Note 9.3. In order to solve the previous problem exactly, we need to solve the equation \( \tan(\theta) = 0.08 \) for the angle \( \theta \). It is possible to estimate a solution using a graph or a table of values, but to find an exact solution, we need an inverse for the tangent function.

Definition 9.4. For right triangles,

- \( \theta = \tan^{-1}(x) \) means that \( \theta \) is the angle between 0° and 90° whose tangent is \( x \)
- \( \theta = \cos^{-1}(x) \) means that \( \theta \) is the angle between 0° and 90° whose cosine is \( x \)
- \( \theta = \sin^{-1}(x) \) means that \( \theta \) is the angle between 0° and 90° whose sine is \( x \)

Notation 9.5.

- \( \tan^{-1}(x) \) is also written arctan(\( x \))
- \( \cos^{-1}(x) \) is also written arccos(\( x \))
- \( \sin^{-1}(x) \) is also written arcsin(\( x \))

Example 9.6. \( \cos^{-1}(0.82) = 34.9 \) means that \( \cos(\text{_________}) = \text{_________} \). In other words, (circle one) 0.82 / 34.9 is the angle, and (circle one) 0.82 / 34.9 is the value of cosine.

Example 9.7. Without your calculator, find

a) \( \sin^{-1}(1) \)

b) \( \cos^{-1}(\frac{\sqrt{2}}{2}) \)

c) \( \sin^{-1}(\frac{1}{2}) \)

d) \( \tan^{-1}(1) \)
Example 9.8. Use your calculator to find
a) \( \sin^{-1}(0.77) \)
b) \( \cos^{-1}(0.2) \)
c) \( \tan^{-1}(3.5) \)

Example 9.9. If you use your calculator to find \( \cos^{-1}(2.7) \), you get an error message. Why?

Example 9.10. Find \( \theta \) and \( \phi \) and \( x \).

Example 9.11. Solve for the angle \( \theta \) in a right triangle if
a) \( 5\sin(2\theta) + 7 = 10 \)
b) \( \sin^2(\theta) + 2\sin(\theta) = 1 \)  
Note: \( \sin^2(\theta) \) means \( (\sin(\theta))^2 \).

**Notation 9.12.** Be careful: \( \sin^{-1}(x) \neq (\sin(x))^{-1} \)  
\( (\sin(x))^{-1} \) means the reciprocal of \( \sin(x) \): \( \frac{1}{\sin(x)} \)  
\( \sin^{-1}(x) \) means the inverse function of \( \sin(x) \)

---

**Definition 9.13.** *Solving* a triangle means finding the measures of all the angles and all the sides.

In one of the previous examples, you solved a right triangle. This section covers some techniques for solving triangles that are not right triangles.

**Note 9.14.** For a right triangle, if we know two sides, we can find the third using the Pythagorean Theorem. In the right angled triangle at left, if we know \( a \) and \( b \), we can find \( c \) because \( c^2 = a^2 + b^2 \).
The middle triangle is not quite a right triangle because its angle is slightly less than 90°. For this triangle,
\[ c'^2 \text{ is (circle one) less than / greater than } a^2 + b^2 \]

The triangle on the right is not a right triangle because its angle is slightly more than 90°. For this triangle,
\[ c''^2 \text{ is (circle one) less than / greater than } a^2 + b^2 \]

**Note 9.15.** The Law of Cosines is a generalization of the Pythagorean Theorem to triangles that are not right triangles. It says that for any triangle,
\[ c^2 = a^2 + b^2 + \text{correction factor} \]

The correction factor depends on the angle opposite side \( c \). More specifically:

**Theorem 9.16.** *(The Law of Cosines)* For any triangle with sides \( a, b, \) and \( c \) and angle \( C \) opposite side \( c \),
\[ c^2 = a^2 + b^2 - 2ab \cos(C) \]

**Note 9.17.** When \( C < 90^\circ \), then \( \cos(C) \) is (circle one) positive / negative, so the correction factor \(-2ab \cos(C)\) is (circle one) positive / negative, which means that \( c^2 \) is slightly (circle one) less / more than \( a^2 + b^2 \). Does this agree with your intuition about the middle triangle above?

When \( C > 90^\circ \), then \( \cos(C) \) is (circle one) positive / negative, so the correction factor \(-2ab \cos(C)\) is (circle one) positive / negative, which means that \( c^2 \) is slightly (circle one) less / more than \( a^2 + b^2 \). Does this agree with your intuition about the triangle on the right above?

When \( C = 90^\circ \), then \( \cos(C) = 0 \), so the correction factor \(-2ab \cos(C)\) is 0. Therefore, means that \( c^2 \) and \( a^2 + b^2 \) are equal. In other words, when \( C = 90^\circ \), the Law of Cosines is just the Pythagorean Theorem.
Proof. (Law of Cosines) Give a reason for each of the following statements to justify the proof of the Law of Cosines. Note: this proof works for a triangle in which all angles are acute (≤ 90°). A slightly different argument is needed to handle obtuse angles (> 90°).

![Diagram of a triangle with labels a, b, c, x, and h]

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + h^2 = b^2$ and $(a - x)^2 + h^2 = c^2$</td>
<td></td>
</tr>
<tr>
<td>$h^2 = b^2 - x^2$ and $h^2 = c^2 - (a - x)^2$</td>
<td></td>
</tr>
<tr>
<td>$b^2 - x^2 = c^2 - (a - x)^2$</td>
<td></td>
</tr>
<tr>
<td>$c^2 = b^2 - x^2 + (a - x)^2$</td>
<td></td>
</tr>
<tr>
<td>$c^2 = a^2 + b^2 - 2ax$</td>
<td></td>
</tr>
<tr>
<td>$\frac{x}{b} = \cos(C)$</td>
<td></td>
</tr>
<tr>
<td>$c^2 = a^2 + b^2 - 2ab \cos(C)$</td>
<td></td>
</tr>
</tbody>
</table>
Example 9.18. (Problem 45 in the textbook) To estimate the width of the archaeological mound, archaeologists place two stakes on opposite ends of the widest point. They set a third stake off to the side, and connect ropes between the stakes as shown. Find the width of the mound.

Notation 9.19. By convention, in a triangle, capital letters denote angles and lower case letters denote sides. Angle $A$ is opposite side $a$, angle $B$ is opposite side $b$, and angle $C$ is opposite side $c$.

The Law of Cosines can be written in three forms:

- $c^2 = a^2 + b^2 - 2ab \cos(C)$
- $a^2 = b^2 + c^2 - 2bc \cos(A)$
- $b^2 + a^2 + c^2 - 2ac \cos(B)$
Example 9.20. Find the angles of this triangle.

![Diagram of a triangle with sides 8, 13, and 15]

Question 9.21. How many variables does the Law of Cosines have in it? ______________
How many sides or angles do you need to know in order to use the Law of Cosines to solve for an additional side or an angle? ______________

Note 9.22. If you know two sides and one angle, you can plug into The Law of Cosines and solve for the third side. If you know all three sides, you can plug into the Law of Cosines and solve for an angle. But if you know two angles and just one side, then the Law of Cosines can’t help you solve for a side. In this case, you need the Law of Sines, instead.

Theorem 9.23. (Law of Sines) For a triangle with angles A, B, and C and opposite sides a, b, and c,

\[
\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}
\]

![Diagram of a triangle with sides a, b, and c]

Note 9.24. You may remember from geometry that in a triangle, bigger sides are opposite bigger angles. Explain why this fact follows from the Law of Sines. In other words, if angle A > angle B, why does it follow that side a > side b?
Proof. (Law of Sines) Give a reason for each of the following statements to justify the proof of one part of the Law of Sines. Note: this proof works for a triangle in which all angles are acute \((\leq 90^\circ)\). A slightly different argument is needed to handle obtuse angles \((> 90^\circ)\).

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin(B) = \frac{h}{a}) and (\sin(A) = \frac{h}{b})</td>
<td></td>
</tr>
<tr>
<td>(h = a \sin(B)) and (h = b \sin(A))</td>
<td></td>
</tr>
<tr>
<td>(a \sin(B) = b \sin(A))</td>
<td></td>
</tr>
<tr>
<td>(\frac{\sin(A)}{a} = \frac{\sin(B)}{b})</td>
<td></td>
</tr>
</tbody>
</table>

Give a similar argument to prove the rest of the Law of Sines, namely, that  
\[
\frac{\sin(B)}{b} = \frac{\sin(C)}{c}
\]
Example 9.25. An areal tram starts at a point \( \frac{1}{2} \) mile from the base of a mountain whose face has a 60° angle. The tram ascends at an angle of 20°. What is the length of the cable?

![Diagram of a triangle with a mountain and a cable, showing angles 60° and 20°, and a distance of 0.5 miles.]

Note 9.26. When using the Law of Sines to find angles, you have to be very careful, because the Law of Sines only gives us the sine of an angle, not the angle, and there are two angles between 0° and 180° that have the same sine. For example, if use the Law of Sines to find that \( \sin(A) = \frac{1}{2} \), we don’t know whether \( A = 30° \) or \( A = 150° \). Often, we can use clues from the problem to eliminate one solution as impossible, but sometimes both solutions are possible, as in the following example.
Example 9.27. (an ambiguous case) Suppose that a triangle has sides \( a \), \( b \), and \( c \) and opposite angles \( A \), \( B \), and \( C \). Suppose that \( a = 8 \) and \( b = 7 \), and \( B = 40^\circ \). Use the Law of Sines to find \( A \), \( C \), and \( c \). You should find two sets of answers. Draw a triangle to correspond to each set of answers.

Example 9.28. (the ambiguous case revisited) Solve the same problem using the Law of Cosines. Do you still get two solutions?
Note 9.29. In geometry, to prove that two triangles are congruent (i.e. have the same shape and size), we generally need three measurements to be equal, like two sides and an angle of one triangle are equal to the two sides and angle of the other triangle. The we can use rules like SSS, ASA, and SAS. Similarly, to solve a triangle, we generally need three pieces of information, such as two sides and an angle or all three sides.

Fill out the following table relating these rules and whether or not it is possible to solve the triangle.

<table>
<thead>
<tr>
<th>If we know this information ...</th>
<th>can we solve the triangle?</th>
<th>Which law can be used first: Law of Cosines or Law of Sines?</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ASA</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SSA</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SAS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AAA</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note 9.30. It’s a pain that the Law of Sines can give two answers when you use it to find angles. Even worse, sometimes one of the two answers is extraneous. When possible, I recommend using the Law of Cosines to find angles, instead. Of course, if you already know two angles, you can find the third just by subtracting from 180°.
10 Class 10 - §8.1 and 8.2 - Radians and Sinusoidal Functions

§8.1 starts here

**Topic 10.1.** Radians are a way to measure angles – an alternative to degrees. Sometimes using radians instead of degrees can make a problem much simpler, especially in Calculus.

**Definition 10.2.** An angle’s measure in *radians* is equal to the arc length that the angle spans in a unit circle.

![Diagram](image)

**Example 10.3.** What is the circumference of a unit circle? ______________

**Example 10.4.** Fill out the following table to get an intuition for converting from degrees to radians.

<table>
<thead>
<tr>
<th>angle in degrees</th>
<th>fraction of a circle</th>
<th>arclength</th>
<th>angle in radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>90°</td>
<td>$\frac{1}{4}$ of a circle</td>
<td>$\frac{1}{4} \times 2\pi = \frac{\pi}{2}$</td>
<td>$\frac{\pi}{2}$</td>
</tr>
<tr>
<td>180°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>270°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>360°</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note 10.5.** In general, converting from degrees to radians can be done like any other kind of unit change.
Example 10.6. To convert 2.5 minutes to seconds, we use the fact that there are 60 seconds in 1 minute:

\[2.5 \text{ minutes} \times \frac{60 \text{ seconds}}{1 \text{ minute}} = 150 \text{ seconds}\]

To convert from degrees to radians, we use the fact that there are \( \pi \) radians in 180 degrees.

Example 10.7. Convert 40° to radians.

\[40 \text{ degrees} \times =\]

Fact 10.8. Conversion factors:

Degrees \( \times \) __________ = Radians

Radians \( \times \) __________ = Degrees

Example 10.9. Fill in the table:

<table>
<thead>
<tr>
<th>degrees</th>
<th>radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>70°</td>
<td></td>
</tr>
<tr>
<td>53°</td>
<td></td>
</tr>
<tr>
<td>( \frac{5\pi}{4} )</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td></td>
</tr>
</tbody>
</table>

Note 10.10. You might wonder, when converting from degrees to radians, when should you write the answer as a fraction times \( \pi \) and when should you write it as a decimal? A good rule of thumb is that if you get an answer that is a simple multiple of \( \pi \), like \( \frac{3\pi}{4} \), then leave it in that form; otherwise, convert it to a decimal.
Example 10.11. On the unit circle, label the indicated “special angles” with their degree and radian measures.
Note 10.12. This applet http://www.purposegames.com/game/angles-of-the-unit-circle-radians-quiz is a good way to practice radians on the unit circle.

Example 10.13. Without your calculator, calculate the following values, where the angles are in radians.

a) \( \cos\left(\frac{\pi}{3}\right) \)

b) \( \sin\left(\frac{2\pi}{3}\right) \)

c) \( \tan(\pi) \)

Example 10.14. Mark these radians on a unit circle:

a) 1 radian

b) 2 radians

c) 3 radians

d) 4 radians

e) 5 radians

f) 6 radians

g) 7 radians

Example 10.15. Find the length of arc cut off by the angle in each picture. Pictures are not necessarily drawn to scale. Remember that the circumference of a circle with radius \( r \) is \( 2\pi r \).

(a) \hspace{2cm} (b)
89
Fact 10.16. The arclength \( s \) spanned by an angle of \( \theta \) radians in a circle of radius \( r \) is given by the formula

\[ s = \text{___________} \]

The arclength \( s \) spanned by an angle of \( z \) degrees in a circle of radius \( r \) is given by the formula

\[ s = \text{___________} \]

Example 10.17. If you run around a circular track of radius 150 meters for an angle of 250°, what distance do you run?

Example 10.18. The track team practices a 400 meter run using a circular track of radius 50 meters. At what angle do they end up relative to their starting position?

Note 10.19. Your calculator can find cos, sin and tan of angles in radians or degrees. Be sure to set the MODE on your calculator to radians if need to calculate cos, sin and tan of an angle in radians. The degree / radian mode only affects the three buttons labeled cos, sin and tan (and the functions cos\(^{-1}\), sin\(^{-1}\) and tan\(^{-1}\) above them); nothing else is affected.
Topic 10.20. This section deals with transformations of the functions sine and cosine, which are also called sinusoidal functions.

Example 10.21. Use your calculator to plot $y = \cos(x)$ and $y = \sin(x)$ in radian mode, and sketch the plots below.

Definition 10.22. A periodic function is a function that repeats at regular horizontal intervals. The horizontal length of the smallest repeating unit is called the period.

Example 10.23. The period of $y = \cos(x)$ is _______________. The period of $y = \sin(x)$ is _______________.

Definition 10.24. The midline of a periodic function is the horizontal line halfway in between the maximum and minimum points.

Example 10.25. The midline of $y = \cos(x)$ is the _______________. The midline of $y = \sin(x)$ is _______________.

Definition 10.26. The amplitude of a periodic function is vertical distance between a maximum point and the midline. Equivalently, the amplitude is the vertical distance between a minimum point and the midline.

Example 10.27. The amplitude of $y = \cos(x)$ is the _______________. The amplitude of $y = \sin(x)$ is _______________.

Note 10.28. The amplitude of sinusoidal functions can also be found by taking half of the vertical distance between a maximum point and a minimum point.

Example 10.29. Find the midline, amplitude, and period for the following graphs:
Note 10.30. The following exercises can be done either on your calculator or on a computer using the corresponding Sage math worksheet.

Example 10.31. For each of the following, graph the functions and observe the changes in midline, amplitude, and/or period.

<table>
<thead>
<tr>
<th>Function</th>
<th>Midline</th>
<th>Amplitude</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = \cos(x)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g(x) = \cos(x) + 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h(x) = \cos(x) - 2$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Question 10.32. How does the graph of $y=\cos(x)+k$ compare to the graph of $y=\cos(x)$?
Function | Midline | Amplitude | Period
--- | --- | --- | ---
$f(x) = \cos(x)$ | | | |
g$(x) = 4 \cdot \cos(x)$ | | | |
h$(x) = 0.5 \cdot \cos(x)$ | | | |

| Function | Midline | Amplitude | Period |
--- | --- | --- | ---
$f(x) = \cos(x)$ | | | |
g$(x) = -\cos(x)$ | | | |
h$(x) = -3 \cdot \cos(x)$ | | | |

**Question 10.33.** How does the graph of $y = A \cdot \cos(x)$ compare to the graph of $y = \cos(x)$?

**Question 10.34.** What effect does it have if $A$ is negative instead of positive?

**Fact 10.35.** The graph of $y = A \cdot \cos(x) + k$ has midline at _____________ and amplitude of ___________. If $A$ is negative, then the graph is ___________ over its midline. The same things hold for the graph of $y = A \cdot \sin(x) + k$.

**Example 10.36.** For each of the following, graph the functions and observe the changes in midline, amplitude, and / or period.
Fact 10.37. The function $y = \sin(B \cdot t)$ and $y = \cos(B \cdot t)$ have periods of ___________.

Example 10.38. Spend a few minutes using the Sage math worksheet to graph sinusoidal functions and determine the equations.

Note 10.39. When finding equations for graphs of sinusoidal functions, there are essentially four functions to keep in mind:

Example 10.40. In the United States, electricity cycles between 155.6 volts and -155.6 volts 60 times per second. Use cosine to model the voltage. Hint: start by drawing a rough graph of electricity over time. Then find the midline, amplitude, period, and equation.
Example 10.41. A Ferris wheel has a diameter of 100 feet. It turns once every 5 minutes. You start at time \( t = 0 \) at the bottom, which is 10 feet above the ground. Write an equation to describe your height above ground as a function of time. Hint: start by drawing a rough graph.

Note 10.42. For the function \( y = A \cos(Bt) + k \), we have seen the following:

<table>
<thead>
<tr>
<th>parameter</th>
<th>transformation</th>
<th>effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>vertical shift</td>
<td>change in midline (midline = ( k ))</td>
</tr>
<tr>
<td>( A )</td>
<td>vertical stretch / shrink</td>
<td>change in amplitude (amplitude = ( ____________ ))</td>
</tr>
<tr>
<td>( B )</td>
<td>horizontal stretch / shrink</td>
<td>change in period (period = ( ____________ ))</td>
</tr>
</tbody>
</table>

The final transformation that we need to study is the horizontal shift.

Recall that for any function \( f(x) \), the graph of \( f(x+c) \) is the graph of \( f(x) \) shifted horizontally by \( c \). For example, \( f(x - 3) \) shifts the graph of \( f(x) \) \( \_\_\_\_\_\_\_\_\_\_\_\_ \) by \( \_\_\_\_\_\_\_\_\_\_\_\_ \).
Example 10.43. Consider the graph \( f(x) = 3 \sin(2x) \) below. Without using your calculator, sketch where the graph of \( g(x) = 3 \sin(2(x - \frac{\pi}{4})) \) should be on the same axes. Hint: \( g(x) = f(x - \frac{\pi}{4}) \). Check your work on your calculator or a computer.

![Graph of \( f(x) = 3 \sin(2x) \) and \( g(x) = 3 \sin(2(x - \frac{\pi}{4})) \)](image)

Conclusion: the graph of \( g(x) = 3 \sin(2(x - \frac{\pi}{4})) \) is the graph of \( f(x) = 3 \sin(2x) \) shifted right by \( \frac{\pi}{4} \).

Warning 10.44. It looks like \( g(x) = 3 \sin(2x - \frac{\pi}{2}) \) might have a shift of \( \frac{\pi}{2} \) but in fact that is not correct. In fact the shift is \( \frac{\pi}{4} \) because \( 3 \sin(2(x - \frac{\pi}{4})) = 3 \sin(2(x - \frac{\pi}{4})) \). You have to factor to find the correct horizontal shift.

Fact 10.45. The graph of \( y = \sin(B(x - h)) \) is the same as the graph of \( y = \sin(Bx) \) shifted right by \( h \).

Example 10.46. The horizontal shift of \( y = \cos(4t + \frac{\pi}{3}) \) is the graph of \( y = \cos(t) \) shifted (circle one) left / right by \( \frac{\pi}{3} \).

Example 10.47. Find the equation of this graph:
Example 10.48. The graph of \( y = \cos(x) \) is the same as the graph of \( y = \sin(x) \) shifted horizontally by \( \frac{\pi}{2} \). In other words

\[
\cos(x) = \sin(x + \frac{\pi}{2}).
\]

Also,

\[
\sin(x) = \cos(x - \frac{\pi}{2}).
\]

**Topic 10.49.** Phase shift versus horizontal shift

**Definition 10.50.** The *phase shift* is defined as \( 2\pi \) times the fraction of a period that the graph is shifted.

**Example 10.51.** In the graph above, the fraction of a period that the graph is shifted is \( \frac{h}{2\pi} \). The phase shift is \( \frac{\pi}{2} \).

**Question 10.52.** In the example above, how is the phase shift related to the horizontal shift and the number \( B \) in the equation for the function?

\[
\text{Phase shift} = B \times h
\]

**Fact 10.53.** In the equation \( y = \sin(B(t - h)) \), or the equivalent equation \( y = \sin(Bt - Bh) \), the horizontal shift is \( h \) and the phase shift is \( \frac{\pi}{2} \).

**Proof.** (Optional)

Fill in the reasons for the following proof that

\[
\text{Phase shift} = B \times h
\]

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>In the equation ( y = \sin(B(t - h)) ), the horizontal shift is ( h ) and the period is ( \frac{2\pi}{B} ).</td>
<td>rules for transformations of functions</td>
</tr>
<tr>
<td>So the fraction of a period that the graph is shifted is ( \frac{h}{\frac{2\pi}{B}} ).</td>
<td></td>
</tr>
<tr>
<td>Therefore, the phase shift is ( 2\pi \times \frac{h}{\frac{2\pi}{B}} ).</td>
<td></td>
</tr>
<tr>
<td>So the phase shift is ( Bh ).</td>
<td></td>
</tr>
</tbody>
</table>

**Example 10.54.** In the function \( y = 5 \cos(\pi x + \frac{\pi}{2}) \) what is the phase shift? \( \frac{\pi}{2} \) What is the horizontal shift? \( h \) What fraction of a period is the function shifted? \( \frac{1}{2} \)
11 Class 11 - §8.3 and 8.4 - Other Trig Fns and Inverse Trig Fns

Definition 11.1. There are three more trig functions:

• $\sec(x) = \frac{1}{\cos(x)}$

• $\csc(x) = \frac{1}{\sin(x)}$

• $\cot(x) = \frac{1}{\tan(x)}$

Note 11.2. $\cot(x)$ can also be written as $\frac{\cos(x)}{\sin(x)}$. Why?

Example 11.3. Find the values (angles are in radians)

a) $\csc(\frac{\pi}{3})$

b) $\sec(2\pi)$

c) $\cot(0.875)$

Example 11.4. Without using your calculator, match the following functions to their graphs. Hint: think about the values of sin and cos at key angles like 0 and $\frac{\pi}{2}$ and then calculate sec and csc at these angles.

• $y = \sec(x)$

• $y = \csc(x)$

• $y = -\sec(x)$

• $y = -\csc(x)$

• $y = \tan(x)$

• $y = \cot(x)$
Fact 11.5. An important relationship between sin($t$) and cos($t$) is the “Pythagorean Identity”:

$$(\sin(t))^2 + (\cos(t))^2 = 1$$

This identity is usually written as:

$$\sin^2(t) + \cos^2(t) = 1$$
because by convention, $\sin^2(t)$ means $(\sin(t))^2$.

Explain why the Pythagorean Identity holds using the unit circle.

Example 11.6. Suppose that $\sin(\alpha) = \frac{5}{13}$ and $\alpha$ is in the first quadrant. Find $\cos(\alpha)$ and $\tan(\alpha)$. Hint: one method is to use the Pythagorean Identity. Another method is to draw a triangle. I encourage you to try out both methods.

Example 11.7. In the previous example, how would your answers change if you were told that $\alpha$ is in the second quadrant instead of the first quadrant?
12.1. We have already used the inverse trig functions \( \sin^{-1}(x) \), \( \cos^{-1}(x) \), and \( \tan^{-1}(x) \) to solve triangles. In this section we will revisit these inverse trig functions and learn some of the theory behind them.

**Example 12.2.** Find all the values of \( t \) for which \( \cos(t) = 0.6 \). Assume that \( t \) is in degrees. You won’t be able to list all values, since there are infinitely many, but list a few and describe the pattern.

Mark the approximate location of the solutions on the unit circle and on the graph.
Example 12.3. Solve $2 \cos(t) = 1 - \cos(t)$, assuming $t$ is in radians.

a) Find all values of $t$

b) Find all values of $t$ between 0 and $2\pi$. How many solutions do you expect between 0 and $2\pi$? ____________

Mark the approximate location of the solutions on the unit circle and on the graph.

Definition 12.4. The solutions of a trigonometric equation which lie in the interval $[0, 2\pi)$ are called principal solutions.

Note 12.5. There are usually two principal solutions, corresponding to two angles on the unit circle. The other solutions can be found by adding and subtracting multiples of $2\pi$. 
Note 12.6. When solving equations involving trig functions, you should assume that angles are given in radians, not degrees, unless otherwise specified.

Example 12.7. Find all solutions to $4 \sin(t) - 1 = 2$ between 0 and $4\pi$. How many should there be? 

Mark the approximate location of the solutions on the unit circle and on the graph.
Example 12.8. a) Find all solutions to \( \tan(t) = 4 \).

b) Find all solutions in the interval \( 0 \leq t \leq 2\pi \) and mark them on the graph.
Fact 12.9. Summary:

1. To find solutions to the equation \( \cos(t) = M \)
   - use \( \cos^{-1}(M) \) to find one principal solution
   - use \( 2\pi - \cos^{-1}(M) \) to find the other principal solution
   - add and subtract multiples of \( 2\pi \) to the two principal solutions to find all other solutions

2. To find solutions to the equation \( \sin(t) = N \)
   - use \( \sin^{-1}(N) \) to find one principal solution (if \( \sin^{-1}(N) \) is negative, add \( 2\pi \) to make it positive)
   - use \( \pi - \sin^{-1}(N) \) to find the other principal solution
   - add and subtract multiples of \( 2\pi \) to the two principal solutions to find all other solutions

3. To find solutions to the equation \( \tan(t) = P \)
   - use \( \tan^{-1}(P) \) to find one principal solution
   - add and subtract multiples of \( \pi \) to the principal solution to find all other solutions

4. In most cases, the equation \( \cos(t) = M \) or \( \sin(t) = N \) or \( \tan(t) = P \) has two solutions per interval of length \( 2\pi \). So for example, there are generally \( \text{______________} \) solutions for \( 0 \leq t \leq 6\pi \).

Example 12.10. Find a value of \( M \) for which the equation \( \cos(x) = M \) has less than two solutions in the interval \([0, 2\pi]\).
**Note 12.11.** We have seen that equations like \( \cos(x) = 0.35 \) have an infinite number of solutions. But the calculator only gives one value for \( \cos^{-1}(0.35) \). How does it pick which one?

Recall that to have an inverse, the graph of the function must satisfy the horizontal line test. Does the graph of \( y = \cos(x) \) satisfy this property? 

Even though \( y = \cos(x) \) doesn’t satisfy the horizontal line test, we can restrict \( y = \cos(x) \) to the largest piece that satisfies the horizontal line test. Shade in a piece of the graph of \( y = \cos(x) \) that does satisfy the horizontal line test. This is our **restricted** \( \sin \) function.

![Graph of \( y = \cos(x) \)](image)

**Definition 12.12.** \( \cos^{-1}(x) \) is the inverse of the restricted cosine function.

<table>
<thead>
<tr>
<th>Restricted ( \cos(x) ) has</th>
<th>( \cos^{-1} ) has</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain:</td>
<td>Domain:</td>
</tr>
<tr>
<td>Range:</td>
<td>Range:</td>
</tr>
</tbody>
</table>

\( \cos^{-1}(x) \) is the angle between \( \_ \_ \_ \_ \_ \_ \) whose cosine is \( x \).

**Example 12.13.** Shade in a piece of the graph of \( y = \sin(x) \) that does satisfy the horizontal line test. This is our **restricted** \( \sin \) function.

![Graph of \( y = \sin(x) \)](image)

**Definition 12.14.** \( \sin^{-1}(x) \) is the inverse of the restricted cosine function.
Restricted \( \sin(x) \) has \( \sin^{-1} \) has

Domain: _____________  Domain: _____________

Range: _____________  Range: _____________

\( \sin^{-1}(x) \) is the angle between _____________ whose sine is \( x \).

**Example 12.15.** Shade in a piece of the graph of \( y = \tan(x) \) that does satisfy the horizontal line test. This is our **restricted** \( \tan \) function.

![Graph of \( y = \tan(x) \)](image)

**Definition 12.16.** \( \tan^{-1}(x) \) is the inverse of the restricted cosine function.

Restricted \( \tan(x) \) has \( \tan^{-1} \) has

Domain: _____________  Domain: _____________

Range: _____________  Range: _____________

\( \tan^{-1}(x) \) is the angle between _____________ whose tangent is \( x \).

**Example 12.17.** What is the difference between solving these two equations?

- \( \cos(t) = -\frac{1}{2} \)
- \( \cos^{-1}(-\frac{1}{2}) = t \)
Example 12.18. Write the trig function (cos, sin, or tan) on each graph and label the parts of the graph with the labels:

- trig function
- restricted trig function
- flip of the trig function
- inverse trig function
13 Class 12 - §9.1 and 9.2 - Trig Identities

Example 13.1. Find solutions to the following equations:
   a) \( x^2 - 6x = 7 \)

   b) \( x^2 - 6x = 7 + (x - 7)(x + 1) \)

Definition 13.2. The second equation is called an identity because it is always true, no matter what value of \( x \) you plug in.

Example 13.3. Decide which of the following equations are identities. Hint: experiment, using your calculator. Test out values of \( \theta \), or draw graphs.
   a) \( \cos(\pi - \theta) = \sin(\theta) \)

   b) \( \cos(\theta + \pi) = -\cos(\theta) \)
c) \( \cos^2(\theta) = 1 - \sin^2(\theta) \)

**Question 13.4.** What are some things that you can do to decide if an equation is an identity?

1.
2.
3.

We have already encountered the most important trig identity: the Pythagorean identity, which says:

**Identity 1:**

\[ \cos^2(\theta) + \sin^2(\theta) = 1 \]

**Example 13.5.** Show that \( \cos^2(\theta) = 1 - \sin^2(\theta) \) and the identity \( \sin^2(\theta) = 1 - \cos^2(\theta) \) follow from the Pythagorean identity by rewriting it.

**Identity 2:** \( \tan^2(\theta) + 1 = \sec^2(\theta) \)

Verify that this is an identity by graphing both sides. Then prove it by dividing both sides of the Pythagorean identity by \( \cos^2(\theta) \) and simplifying.
Identity 3: \( \cot^2(\theta) + 1 = \csc^2(\theta) \)

Prove this identity by dividing both sides of the Pythagorean identity by \( \sin^2(\theta) \) and simplifying.

Identity 4: \( \cos(-t) = \cos(t) \)

Explain why this identity is true using the unit circle or a graph of \( y = \cos(t) \).

Identity 5: \( \sin(-t) = -\sin(t) \)

Explain why this identity is true using the unit circle or a graph of \( y = \sin(t) \).
Topic 13.6. Identities are useful for simplifying expressions and solving equations.

Example 13.7. Simplify
\[
\frac{\sin^2(t)}{1 - \cos^2(t)}
\]

Hint: use the Pythagorean identity to rewrite \(\sin^2(t)\).

Example 13.8. Solve for \(t\), for \(0 \leq t \leq 2\pi\)
\[
\cos^2(t) = 3 - 3 \sin(t)
\]
Identity 6 and 7: The angle sum formulas:

\[
\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B) \\
\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)
\]

Identity 8 and 9: The angle difference formulas:

\[
\sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B) \\
\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)
\]

See if you can figure out the proof of these formulas from the pictures.

Identity 10: Double angle formula for sine:

\[
\sin(2\theta) = 2\sin(\theta)\cos(\theta)
\]

Prove the double angle formula for sine from the angle sum formula for sine. Hint: plug in \(\theta\) for \(A\) and \(B\).
Identity 11: There are three forms for the double angle formula for cosine.

\[
\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \\
\cos(2\theta) = 1 - 2\sin^2(\theta) \\
\cos(2\theta) = 2\cos^2(\theta) - 1
\]

Prove the first double angle formula for cosine by using the angle sum formula for cosine.

Prove the other two formulas from the first one using the Pythagorean identity.

Example 13.9. Solve \(\sin(2t) = \sin(t)\)
Example 13.10. Simplify
\[ \frac{\cos(t + \frac{\pi}{2})}{\cos(t - \pi)} \]

Example 13.11. Use the angle sum and difference formulas to find exact values for \( \cos(105^\circ) \) and \( \sin(105^\circ) \). Hint: write 105° as a sum or difference of two special angles.

Identity 12: Half angle formulas:
\[
\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos(\theta)}{2}} \\
\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{2}}
\]

(Optional) Prove the half angle formulas using the double angle formulas for cosine. Hint: \( \theta = 2 \cdot \frac{\theta}{2} \), so plug in \( \frac{\theta}{2} \) into the double angle formula.

**Example 13.12.** Suppose \( \theta \) is an angle between 0 and \( \frac{\pi}{2} \) and \( \cos(\theta) = \frac{4}{y} \) for some number \( y \).

a. Find \( \sin(\theta) \) in terms of \( y \).

b. Find \( \cos(2\theta) \) in terms of \( y \).

c. Find \( \cos\left(\frac{\theta}{2}\right) \) in terms of \( y \).